Wasit University
Computer Sciences and Information Technology College

Wasit Journal
OF COMPUTER AND MATHEMATICS

A Quarterly Academic Refereed Journal
Issued by Computer Sciences and Information Technology College

Wasit Journal of computer and Mathematics

Issue 2 Volume 1

Editor by

Prof.Dr Ali Khalaf Hussain, Wasit University, Iraq

https://wjcm.uowasit.edu.iq/
Additional copies can be obtained from:
wjcm@uowasit.edu.iq
published April 2022
Printed in Iraq
Address: University of Wasit ,hay alrabee, Kut, Wasti, Iraq

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Finger Vein Recognition Using Deep Learning Technique

https://doi.org/10.31185/wjcm.Vol1.Iss2. 43

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Abstract—Due to their combination of security and economic viability, finger vein biometrics have gained considerable traction in recent years. They have the advantage of being the least vulnerable to identity theft because veins are present beneath the skin, as well as being unaffected by the ageing process of the user. To address the ever-increasing need for security, all of these variables necessitate working models. Using face recognition and AI-based biometrics has become a hot subject in law enforcement because of the accidental demographic bias it introduces into the process. Biometric prejudice, on the other hand, has far-reaching implications that transcend into everyday situations. When an ATM transaction or an online banking transaction is compromised by a fake positive or negative verification, it makes it simpler for fraudsters to carry out their criminal activities. The veins of a fingertip were the subject of this research project's investigation. Deep convolutional neural network models were utilised to extract features from two widely-used and freely-available datasets of finger veins. Finger vein identification as a unique biometric approach has received a lot of attention recently. Accuracy of greater than 98 percent is reached with the deployment of multi-class categorization. The binary classification based model has a 97.51 percent accuracy rate. The total outcomes and their effectiveness are fairly good with the implementation situations. Deep learning, an end-to-end technique that has demonstrated promising results in domains like face recognition and target detection, may be useful for finger vein recognition.

Keywords—Deep Learning, Finger Vein Detection, Finger Vein Analysis using Deep Learning

1 Introduction

To authenticate a person's identification, a biometric authentication system measures certain physical traits or behaviors of the person's body in real-time. Iris scanners, for example, use biometric data to create digital representations of a person's identity. Accordingly, biometric authentication systems may identify or validate the individual by comparing the data to other biometric records in the database using algorithms. Biometric authentication systems have two primary modes of operation: identification and verification. The input data is compared to all known patterns in the database in the identification mode. It is possible to determine if this individual is in the database using the system. Biometric input data is compared to a single person's unique pattern when in verification mode. If they're not the same person, it's a way to stop many individuals from using the same identity [1-2].

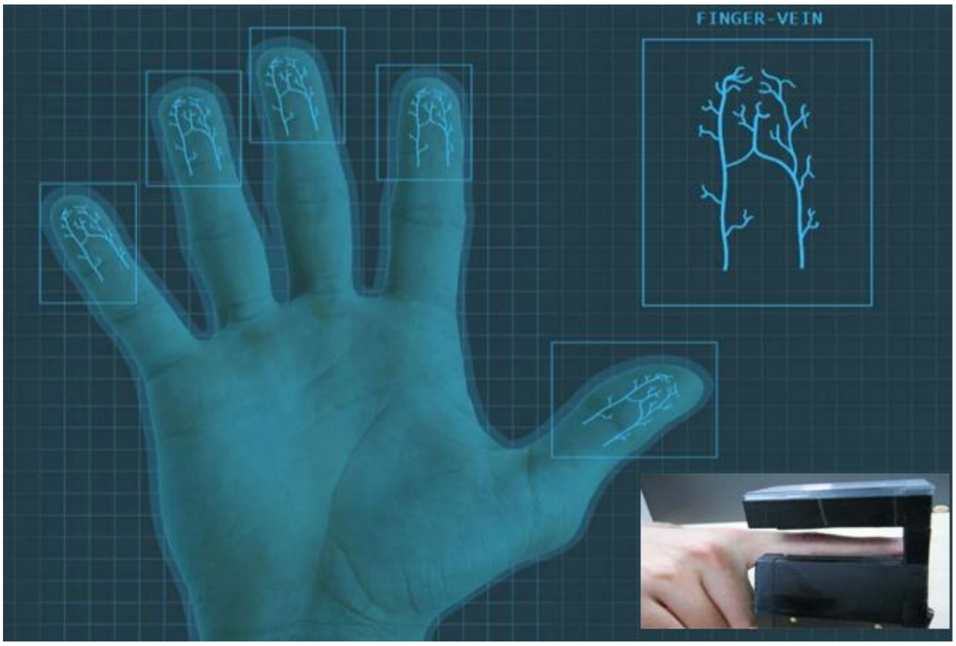

Fig. 1. Finger Vein Recognition Patterns.

Biometric identification systems, like those shown in movies and science fiction, can be deceived by phoney resources. The Hassassin in Dan Brown's novel "Angels & Demons" chopped off Leonardo's eye to steal the antimatter that was locked behind a door with retina scanners [3-4]. Retinal fingerprints are unique to each individual, yet hackers can still find a method to get around them. The amount of protection provided by various biometric features also varies. It is more difficult to deceive the finger-vein based biometric verification system that the work is discussing today since it only recognizes the unique patterns of finger-veins beneath the skin of the live individual [5-6].

Key Points in Finger Vein and Research Statement

Finger-vein data is gathered with the use of specialized capturing equipment. Near-infrared light, a lens, a light filter, and picture capturing technology make up the bulk of this capture apparatus. Finger veins are invisible to the naked eye because they are hidden beneath the surface of the skin. Near-infrared light, which may penetrate through human tissue, is used in this gadget [7-8]. Near-infrared light is also blocked by pigments like hemoglobin and melanin [9].

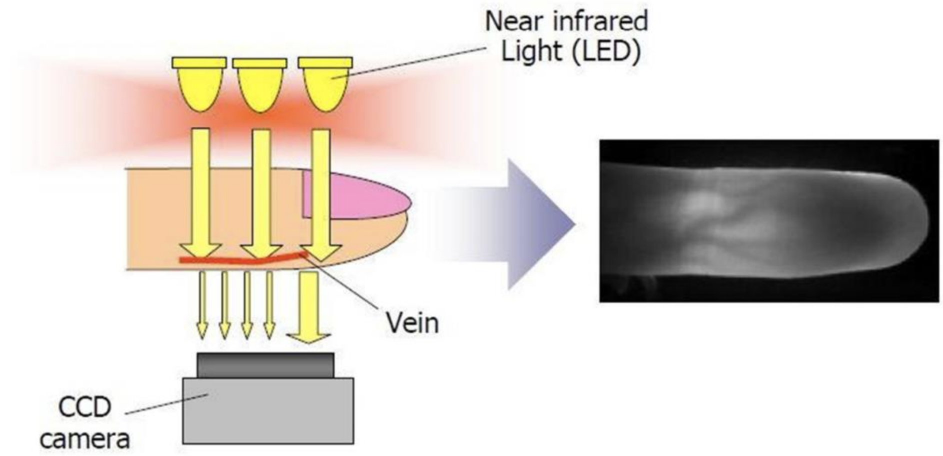

Fig. 2. Finger Vein Data Capturing

Finger vein biometrics uses a person's individual vein patterns to determine their identity. Biometrics derived from the blood vessels beneath the skin is also known as vascular biometrics. Using near-infrared light or visible light causes haemoglobin — the iron-containing protein all have in our blood — to change colour. This means that the reader is able to scan the vein patterns of the individual user. In the cloud, the vein pattern is kept in an encrypted digital format [10].

A near-infrared reader device has been in use for more than a decade for access control systems (people entering a building) and ATMs (people withdrawing money). To identify people on internet services, finger vein biometrics proved its versatility when the world became more connected than it is today [11].

Finger vein biometrics can be used as a primary or a secondary method of authentication for internet services. This is an example of multi-factor authentication (MFA), which involves two factors: a password or social media login (the first factor), and a finger vein authentication (the second factor) [12-14].

Table 1. Modalities with Biometrics

Biometrics	Long-term Stability	Data Size	Cost	Accuracy	Security Level
Finger Vein	High	Medium	High	High	High
Fingerprint	Low	Small	Low	Medium	Low
Face	Low	Large	High	Low	Low
Iris	Medium	Large	High	High	Medium
Voice	Low	Small	Medium	Low	Low
Hand Geometry	Low	Large	High	Low	Low

2 Research Aspects and Methodology:

To recognize photos of 10 handwritten digits, the earliest neural networks LeNet were used, but nowadays, convolutional neural networks (CNNs) are widely renowned for their ability to categorize 1000 images in the ImageNet database. Conventional computer vision algorithms are often outperformed by CNNs because they are excellent at automatically extracting characteristics from pictures [15].

Consider finger-vein identification as an image classification issue. The use of CNNs to solve the finger-vein identification problem must be intriguing! To meet the demands of biometric authentication systems, what kind of tests should conduct? It is common practice to first extract features, and then utilize that information to determine the distance between them. The distribution of feature distances is used to establish a threshold. If the gap between the two traits is more than a certain threshold, they are not considered to be the work of the same author. As long as the gap between these two traits is less than the threshold, they are classified as belonging to the same individual [16-20].

Fig. 3. Interior Analytics on Finger Veins

Finger-vein databases are available from a number of research institutions. SDUMLA-HMT has provided us with the finger-vein dataset that is utilised.

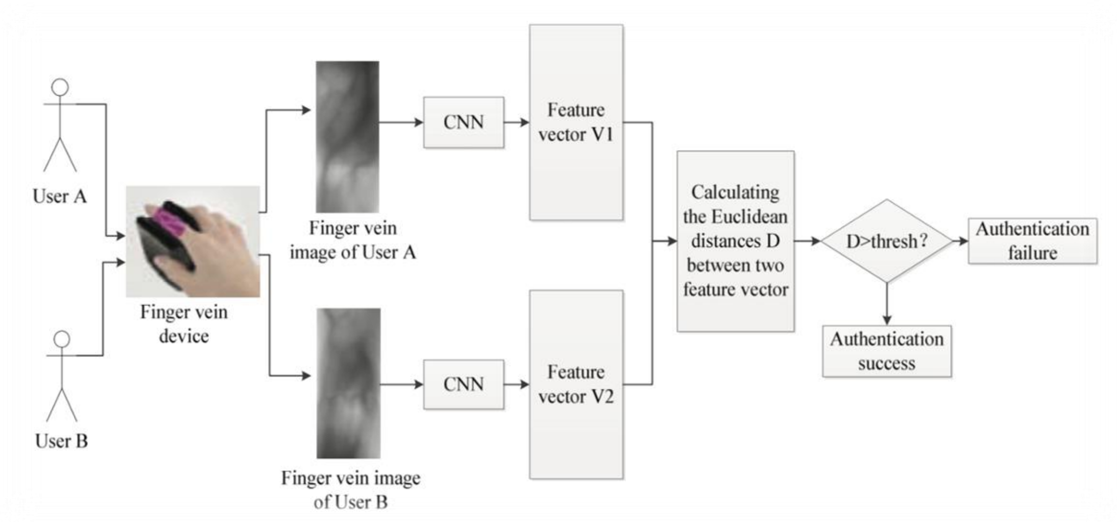

Fig. 4. Analytics Patterns and Methodology.

Here would like to thank Shandong University's MLA Lab for creating the SDUMLA-HMT Database for the work [37]. 106 persons had their finger-vein pictures recorded in this collection. Index, middle and ring fingers of both hands were snatched by the thugs. There are six images on each finger. The total number of photos is 3,816. Images are in the "bmp" format and have a resolution of 320x240 pixels.

Image capture includes not only a person's finger, but also the background, which is often a camera. In order to preserve the finger portion of the image and eliminate the backdrop, it is necessary to extract ROI. It's necessary to determine the ROI's upper and lower bounds [21-22].

Finger-vein Detection using Transfer Learning:

Large datasets need the use of CNN models with a large number of parameters. Complex CNN models need a long time and a lot of resources to train from the beginning. To begin training from scratch in most circumstances, there is not enough data. Another factor to consider is 'overfitting. Insufficient data and a sophisticated model both increase the risk of model overfitting [23, 24].

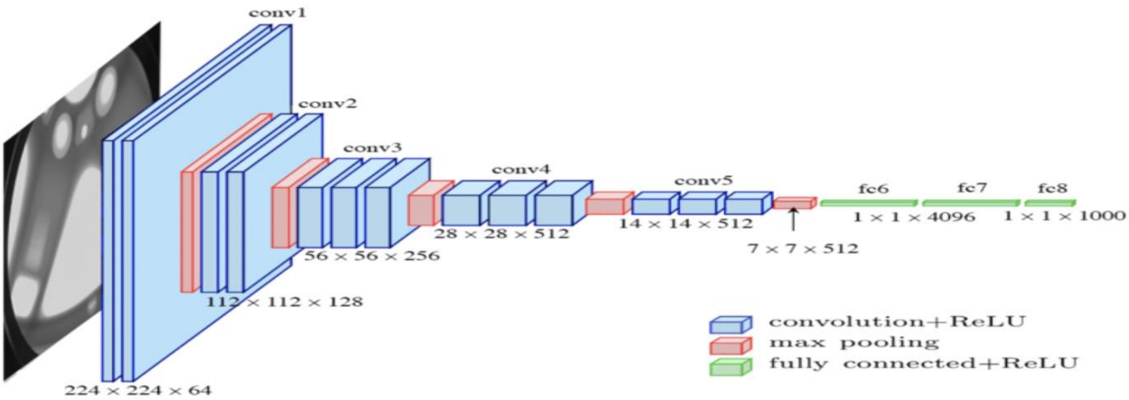

Fig. 5. Architecture of Deep Learning Model for Finger Vein Recognition.

This challenge can be solved with the help of transfer learning [25-26]. The goal of Transfer Learning is to use an existing model (one that has been pre-trained on a significant quantity of data) in a new context. Using a model pre-trained on a big dataset of cats and dogs, that can categorize elephants and monkeys or cartoon cats and dogs, for example.

A pre-trained model may not perform as well when applied to other domains or tasks because it was trained without input from the original one. There are often two options available to us. It's possible to think of the pre-trained CNN as an extractor of features. Extracted features can be used as input for a linear classifier. The fine-tuning approach, on the other hand, is frequently used to fine-tune certain high-level layers [27-33]. The early layers of features are more general in nature. However, the more detailed information from the original datasets may be found in the following layers. It's possible that freezing the earliest layers may yield characteristics that can be applied to a wide range of jobs. The work may produce even more specific characteristics in our datasets by fine-tuning the next layers [34-40].

3 Results and Outcomes

In our investigation, we used two alternative models. In a multi-class classification model, the first model is used to identify the second model is a binary classification model used to verify the results The binary classification model uses photographs of differences as inputs. Both models have been fine-tuned using a pre-trained VGG-16 model. In these models, ROIs are not preprocessed. Experiments using ROI data models were also conducted, however the results were not as anticipated. Low-quality datasets and unconfirmed image quality make it impossible to reliably classify classes.

Model Type	Achieved Accuracy Levels
Multi-Class Classification	98.12 %
Binary Classification	97.51 %

The projected approach with multi-class classification is quite effective and giving better results as compared to the classical binary classification.

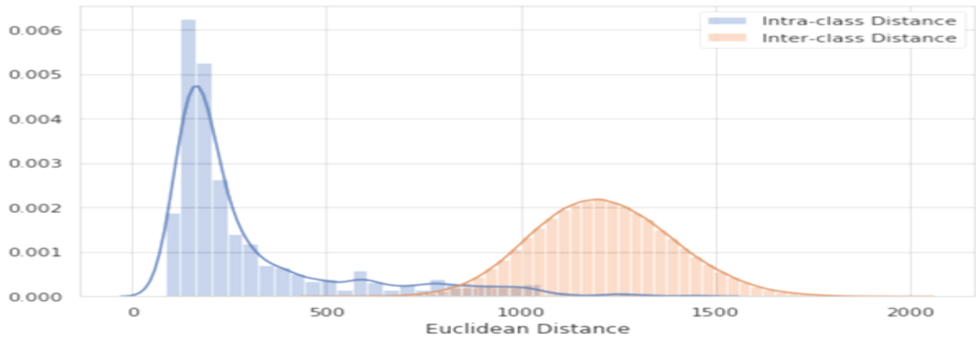

Fig. 6. Performance Evaluation of the Projected Approach.

A model that uses Euclidean distance as the distance between features will provide results that are difficult to explain. Differential images of the same and dissimilar classes, as well as interclass and intraclass distances, are employed in this study. How to describe the distance between two images in simple terms might be a challenge. It's helpful to take photos of the differences between two photographs so you can see just how big the disparity is. Using the Euclidean distance between feature sites as a measurement is unnecessary in this scenario. So, in the binary classification problem, it is possible to calculate FAR and FRR with relative ease According to how many samples match, FRR (false negative) and FAR (false positive) are defined. The FAR and FRR were calculated based on the forecast's probability. A very low error rate (ERR).

Deep learning, as all know, is data-driven. The quality of the data is critical to the success of an experiment. The results are encouraging, despite the poor quality of the dataset utilised. Paper-referenced findings were not met by either the identification or verification models' final outcomes. When compared to other models, deep learning approaches are easier to apply and do not need extensive feature handling and engineering. According to some previous research, still have room for improvement when it comes to pre-processing data, building models, and selecting hyper-parameters.

4 Conclusion

Security has grown increasingly crucial in recent years. The Finger Vein Authentication System has attracted our interest due to its robustness, consistency, and high level of performance. Biometrics, such as fingerprint and iris biometrics, have a lower level of reliability. Finger vein authentication removes the possibility of tampering since it relies on the fact that each person's veins are distinct, even if they are identical twins, and reside beneath the skin their whole lives. In recent years, a number of deep learning algorithms have greatly increased the ability to recognize finger vein patterns. Finger vein authentication and the deep learning approaches used to build the Finger Vein Recognition system are the major objectives of this manuscript.

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Article submitted 2 January 2022. Published as resubmitted by the authors 1 August 2022.

A Novel Arabic Words Recognition System Using Hyperplane Classifier

https://doi.org/10.31185/wjcm.Vol1.Iss2. 45

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Abstract—Topic of exhaustive study for about past decades has been carried out in machine imitation of human reading. a small number of investigates have been accepted on the detection of cursive font writing like Arabic texts for its individual challenge and difficulty .In this work, a novel technique for automatic Arabic font recognition is proposed to demonstrate an suitable recognition rate for multi fonts styles and multi sizes of Arabic word images.The scheme can be classified into a number of steps. First, segmenting Arabic line into words depending on the vertical projection and dynamic threshold then we implicated each Arabic word as a class by ignoring segmenting the word into characters .Second ,normalizing step, the size of Arabic word images varies from each other .The system converts the images that contribution into a new size that is divisible by "N" without remainder, to decrease the difficulty of feature extraction and recognition of the system that may allow images from different resources, Third, feature extraction step which is based on apply the ratio of vertical sliding strips as a features. Finally, multi class support vector machine (one versus one technique)is used as a classifier .This method was estimated on off line printed fonts, five Arabic fonts, (Andalus, Arial, Simplified Arabic, Tahoma and Traditional Arabic) were used and the average recognition rate of all fonts was 95.744%.

Keywords—Support vector machine, one-against-one SVMs technique, vertical projection profile, horizontal projection profile

1 Introduction

Arabic text recognition is an important and demanding task not only for those who speak Arabic but also for non-Arabic national such as Persian and Urdu that use Arabic characters in his language. The most important characteristics of Arabic language can be summarized (Fig.1).

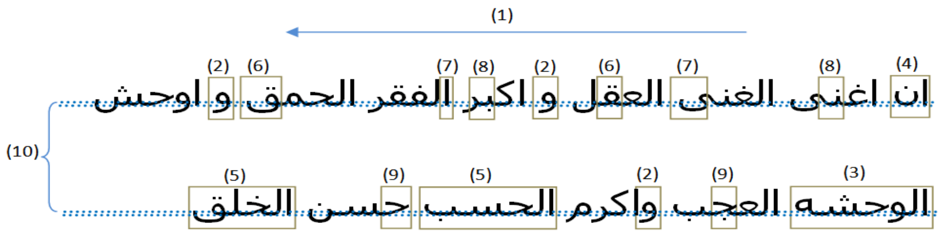

Fig. 1. Characteristics of Arabic text. (1) Direction of Arabic writing from “right” to “left”. (2) Some characters are not connectable from the left side with the succeeding character. (3) A word consisting of six characters. (4) A word consisting of two separate characters. (5) A word consisting of the same number of characters but the word have different size (6) The same character with different shapes depends on its position in the word.(7) Different characters with different sizes. (8) Different characters with a same numbers of dots but different position upper and lower the base line. (9) Different characters withwith a different number of dots. (10) Basie line.

a number of associated work focus on both machine-print and handwriting, with much more discussion of machine-print (B. Al-Badr and S. Mahmoud, 1995 ;M.S. Khorsheed,2002) and hand written (Liana M. Lorigo, And Venu Govindaraju 2006), many methods had been created to establish good and useful Arabic OCR systems. Some of these schemes can be briefly established as follows.

(M.S, Khorsheed,2007)developed a system was based on HMM Toolkit to recognize multi-font Arabic text. (Mehmmood Abdulla Abd, 2007) introduced system to recognize printed Arabic character recognition and utilizing support vectors machine SVMs using one-against-all technique in the classification phase. (Menasri et al.,2007) extracted seventy-four baseline-dependant feature vectors from the graphemes and hybrid HMM/NN to recognize handwritten Arabic words.

(Husni A. et al.,2008) described system for recognition printed Arabic text by applying hierarchical sliding window.

(Jakob Sternbya,2009) explored the application of a template matching scheme to the recognition of Arabic script. (Sami Ben Moussa et al.,2010) proposed method to recognize Arabic font, using global texture analysis based on fractal geometry. (Morteza Zahedi,2011) has proposed a new method for Farsi/Arabic automatic font recognition. (khalifa and yang bing Ru,2011) have computed the Euclidean distance between pairs of objects in n-by-m data matrix X based on the point 's operator of extrema and classify printed and handwritten Arabic words using one against one class SVM.

The intention of this paper aimed to examination problem of automatic recognition Arabic script. Scheming and applying a recognition method of printed Arabic text to answer these issues.

2 ARABIC TEXT SEGMENTATION

Most of the Eliminating unnecessary area: generally there are various useless regions which have no contribution to the recognition. Here, in this section talks about the eliminating unnecessary margin empty region by clipping the word image to get the regains which just consist of the text, line and word.

The image transformed into binary format (0,1) throughout white text with black background, then the row and column scan statistical analysis was done to count the numbers of white pixels showed in horizontal and vertical directions, in that order. To remove the top, bottom, left and right boundary in the same way. Here we use the actual boundaries of the word to extract the crop image (Fig. 2).

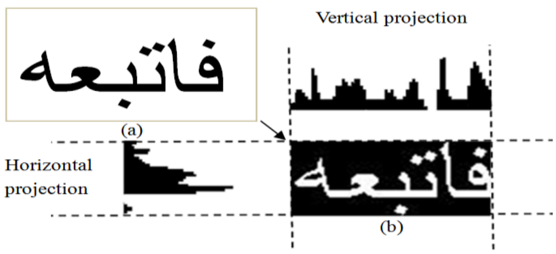

Fig. 2. Crop Arabic word (a) to get new image (b) by removing unwanted region using vertical and horizontal projection profile

Word Segmentation: Because 1)Arabic characters in a word/sub-word are associated at one side (right side) and 2)Arabic words might be consisted of one or more sub-words, there is a distance between two sub-words, in addition to between two words. on the other hand, the space between two words is typically larger than the space between two sub-words.

Thus, segmenting line into words depends on the technique of vertical projection shape that computes the number of the black pixels using the Eq. of the vector Vproj given by:

$$\mathrm{vproj}[i] = \sum_{i=0}^{m-1} I[i,j]$$

where I is a given image text, m is number of rows in a given text.

If the summation of Vprojis equal to zero, that means the gap has started, and we need to calculate the distance of each gap to find the value of the dynamic threshold (T) depending on the mean equation. If the number of zeros is < T that means the line does not segment, or else segments it. The example of product of this method shows in (Fig. 3).

Fig. 3. Example of applying Vertical Projection to segment Arabic line to the words

3 FEATURE EXTRACTION

Printed word has a geometrical shape and this shape is regular whatever the size of the word is modify, in this case, we employed this feature to extract a set of statistical features to get a unique representation to the printed Arabic words images in different sizes. The direction of the Arabic word from right to left is believed as the feature extraction axis. As shown in these steps.

Step1: Pre-processing. A word image is binarazed into white with black background to get a binarazation image and then clip the word image by remove unnecessary area.

Step2: Size of Arabic word images are differ from each other. In order to decrease the difficulty of feature extraction and recognition procedure of this system by accepting image file from different sources, the system implements the action of normalization through transforms different sized image file into a new size by making the width of the image file is divisible by "N" without remainder, where N in this research = 72 then the size of Arabic word image is (LW x WW), where LW is the length of the word image and the WW is the width of the word image.

Step 3: Divided the new image into 72 vertical strips. The length of each slide strip window is the same as the word length(LW) and the width of the slide strip window is (WW/72).

Step 4: Then extract feature from each strip by using the summation of all pixels in each strip as element in one vector named B.

Depend on technique of vertical projection profile, by calculating the number of the white pixels in each strips using the equation of the vector Vproj as shown in this equation.

$$\mathbf{vproj}[\mathbf{i}] = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{m}-\mathbf{1}} \mathbf{I}[\mathbf{i},\mathbf{j}]$$

Where m= number of the rows in the strip

Step 5: From vector B we generate a new vector named C that has 36 elements by compute the ratio between the contiguous elements of vector B after added 2 to the numerator and 1 to the denominator to avoid divide by zero as shown in these equations:

$$C(1) = ((B(1)+2) / (B(2)+1)) * 10 \quad , \quad C(2) = ((B(3)+2) / (B(4)+1)) * 10$$

$$C(3) = ((B(5)+2) / (B(6)+1)) * 10 \ldots C(36) = ((B(71)+2) / (B(72)+1)) * 10$$

4 SUPPORT VECTOR MACHINE

The SVMs consider as a type of hyperplane classifier, it developed based on the statistical learning theory of (V. Vapnik.,1995) , This classifier associated to the error bound of generalization, since it focus on maximizing a geometric margin of hyperplane. The study of SVMs has reported from the mid-1990s,

It wins popularity due to many good features and definitely performance in the fields of nonlinear and high dimensional pattern recognition, but the application of SVMs related to pattern recognition has still continues.SVM classifier in general is a binary (two-class) linear classifier and use kernel function to represents the inner product of two vectors in linear/nonlinear feature space. And it is not straightforward to turn them into multi-class (N-class) recognition systems. There have been many references for describing the details of SVMs, like N. (Cristianini and J. Shawe-Taylor,2000; B. Sch¨olkopf and A. J. Smola,2002; Christopher J.C., 2005).

The implementation of SVM in the proposed system: The features of off-line Arabic word has been extracted from a given word images as discussed previously. The feature vector has feed as a row in one matrix to create one model for all words, that have all feature vectors together and each one of these vectors have been labeled to distinguish one class from the others . In general, classification task usually involves training and testing sets (Fig.4) which consist of data (class labels and a feature vectors).The task of recognition allocates each word (class) within predefined matrix that has all feature vectors. In this work an optimized support vector machines (SVMs) to classify the input Arabic words by applying one-against-one (1-v-1) SVMs technique.

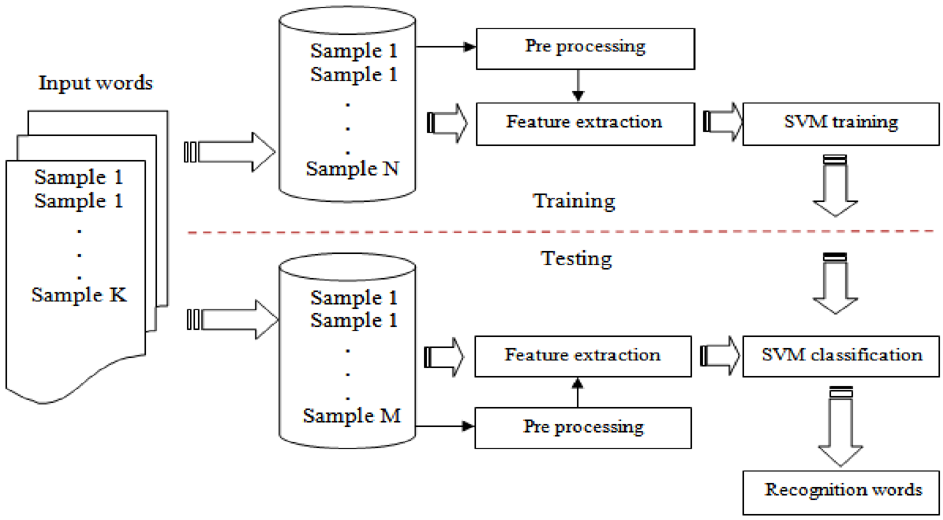

Fig. 4. The main architecture of implementation SVM in the proposed systems

5 EXPERIMENT RESULTS AND ANALYSIS

Experimental datasets: The proposed system has been tested on the printed Arabic text using public dataset (PATS-A01) . We prepare five experimental datasets for analysis based on this database. This dataset named printed Arabic Text Set A01 (PATS-A01) created by(Husni A. et al.,2008). It consists 2766 text line images the line images are available in eight fonts: (Arial, Tahoma, Akhbar, Thuluth, Naskh, Simplified Arabic, Andalus, and Traditional Arabic).

Results and analysis of word segmentation: This section evaluates the methods for segmenting lines into words as previously described. The lines that are used in this experiment have been selected randomly from PATS-A01 dataset, each line is typed in five Arabic fonts(Andalus, Arial, Simplified Arabic, Tahoma and Traditional Arabic). It works well on all fonts with a segmentation rate ofabout 100%. (Table 1) shows segmentation rates and (Table 2) shows some examples of successful segmentation results.

Table 1. Segmentation rates for each fonts

Font name	**Segmentation rate for five fonts from PATS-A01 dataset**
Andalus	100 %
Arial	100 %
Simplified Arabic	100 %

Tahoma	98.485 %
Traditional Arabic	96.97 %
Average	99.091 %

Table 2. Shows some examples of successful segmentation results

Font name	Arabic line samples	Success/ Failure
Andalus	عزة كأب جنة طفق ميس غضبى كثف ضغط أص فظل حذف	yes
Arial	عزة كأب جنة طفق ميس غضبى كثف ضغط أص فظل حذف	yes
Simplified	عزة كأب جنة طفق ميس غضبى كثف ضغط أص فظل حذف	yes
Tahoma	عزة كأب جنة طفق ميس غضبى كثف ضغط أص فظل حذف	yes
Traditional	عزة كأب جنة طفق ميس غضبى كثف ضغط أص فظل حذف	yes

Results and analysis of word recognition: The proposed recognition method has been evaluated on printed Arabic words, five different Arabic fonts were used (Andalus, Arial, Simplified Arabic, Tahoma and Traditional Arabic).

6 Test the performance of all features on each font separately with different size:

This experiment has been applied on (dataset 1, dataset 2 , dataset 3, dataset 4, dataset 5) respectively, Totally each one of the dataset has 396 Arabic words and these words are divided into 66 classes of Arabic words. And the samples in each dataset are typed in one of five different fonts (Andalus, Arial, Simplified Arabic, Tahoma and Traditional Arabic) correspondingly. Therefore to distinguish between these words, the size of each Arabic word has been changed. We will simply resize the model size in each class according to original size as follows:

Model1 = original image size

Model2 = 1.2(original image size)*
Model3 = 1.4(original image size)*
Model4 = 1.6(original image size)*

Table 3. The recognition rate (%) of each fonts

Dataset name	The font name that used in the dataset	Recognition rate %	Approximate recognized samples	Approximate unrecognized samples
Dataset1	**Andalus**	**95.799**	**379**	**17**
Dataset 2	**Arial**	**95.845**	**380**	**16**
Dataset 3	**Simplified Arabic**	**96.183**	**381**	**15**
Dataset 4	**Tahoma**	**95.356**	**378**	**18**
Dataset 5	**Traditional Arabic**	**95.539**	**378**	**18**
	The Average	**95.7444**	**379**	**17**

Model5 = 1.8*(original image size)
Model6 = 2*(original image size)

When the words carried to the system, 36 features extract from each word. It was able to be recognized from each dataset(379 ,380 , 381 ,378 ,378 and 379) approximate models and produced recognition rate (95.799%,95.845%,96.183%,95.356% and 95.539%) correspondingly. See (Table 3).

7 CONCLUSION

Due to specific characteristics of Arabic scripts, such as cursiveness, recognition of Arabic words is considerable more complex than the recognition of English words.

We have proposed a novel scheme for automatic Arabic font recognition which is based on concern vertical sliding strips summation of the words images, then the ratio between two adjacent slides is considered as a feature, this feature feed to the words models which consist of word class labels and word class features. at last, the words models fetched to the classification stage using multi class support vector machine as a classifier by using 1-v-1 technique.

When the words input to the system, 36 features extracted from each word and the average recognition rate of all fonts was 95.744%.

The results obtained are extremely hopeful and have shown that the system work well and fast on off printed Arabic words and can be employ this system in the future to handwritten Arabic words. Also can be test the system on large scale date set .

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Article submitted 15 January 2022. Published as resubmitted by the authors 1 August 2022.

Dynamic Cryptography Integrated Secured Decentralized Applications with Blockchain Programming

https://doi.org/10.31185/wjcm.Vol1.Iss2.41

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Abstract— Abstract Blocks and chains are the building blocks of the blockchain, which is a community network. Blocks and chains are two terms used to describe collections of data information. The most fundamental need for a blockchain is that these postings be connected by cryptography, which is the case here. Cryptography. the entries in each block are added to as the list grows. Although the concept of blockchain cryptography is difficult, we have made it easier for you to understand. Asymmetric-key cryptography and hash functions are used in blockchains. Hash functions provide participants with a complete image of the internet. The SHA-256 hashing algorithm is often used in blockchains. In Bitcoin, where addresses are tracked by public-private key pairs, blockchains are often used. The public key in blockchain cryptography is a person's address. All participants have access to the participant's public key. The private key is used to get access to the address database and to authorise activities using the address. To ensure the integrity of the blockchain ledger, encryption plays a key role. Each event on the blockchain is recorded using encrypted data. As long as each user has access to their cryptographic keys, they may buy or trade cryptocurrencies. The root hashes of all transactions are stored in blockchains via cryptographic hashing. If somebody attempts to tamper with any data upon that blockchain, the main hash will have a completely new hash. Root hash comparisons may be performed on any other system to check whether the data is safe.

Keywords—Blockchain, Dynamic Cryptography, Security in Blockchain

1 Introduction

As a leading technology, blockchain is usually linked with high levels of security and anonymity across a wide range of applications. Blockchain technology is now being used in a wide range of social and business contexts, not only in the cryptocurrency industry. E-governance, social networking and e-commerce are just a few of the many sub-segments that fall under this umbrella [1].

Secured Digital Ledgers with Cryptography in Blockchain

Digital ledgers are stored on a high-security, high-performance system known as a "blockchain." Using a digital ledger eliminates the need for intermediaries or administrators to manipulate the records [2]. All operations on the bitcoin blockchain are finalised using a variety of protocols and processes that cannot be hacked by external parties. [3].

Key Implementations of Blockchain Technology

- Entertainment
  - Spotify
  - Guts
  - B2Expand
  - KickCity
  - Veredictum

- Social Networks
  - Matchpool
  - MeWe
  - Minds
  - Steepshot
  - Mastodon
  - DTube
  - Sola

- Retail
  - Opskins
  - Loyyal
  - Warranteer
  - Every.Shop
  - Blockpoint
  - Fluz Fluz
  - Spl.yt
  - Shopin
  - Ecoinmerce.io
  - Portion
  - Buying.com

- Cryptocurrency
  - Litecoin
  - Ripple

- Primecoin
- Bitcoin
- Namecoin
- Dogecoin
- Nxt
- Ethereum

2 Real Problem and Key Statement

In traditional centralized environment, there is less security because of weak hashing and needs dynamic hash with cryptography as in Blockchain. Transactions cannot be reversed because cryptographic hashing is irreversible. This assures that all users can rely on the digital ledger's correctness and that they are protected from any antagonistic conduct [4, 5].

To understand blockchain, one must understand cryptography. It is possible to encrypt, transmit bitcoin securely, and record transactions over time thanks to cryptography's features. Without a central authority, we may trade bitcoin safely and assure that blocks will continue to be inserted into the chain without restriction [6].

Cryptographic hashing enables blockchains to store vast quantities of transactions while protecting them from hackers. It provides a secure, verifiable, and scalable method for conducting online transactions. Blockchain is genuinely unstoppable because of cryptography [7, 8].

2.1 Public and Private Blockchain with Security and Cryptography

The data structure of blockchain technology is automatically safe. Encryption and decentralisation are the foundations upon which it is based to ensure transactional trust. In most blockchains and DLT, each block of data contains a single transaction or a collection of related transactions. It is extremely impossible to tamper with a cryptographic chain since each new block is irrevocably tied to the blocks that came before it. To ensure that each transaction contained in a block is correct, each transaction must be verified and agreed upon by the consensus process [9, 10].

Blockchain technology is able to transform society since it relies on the participation of all users in a network [11, 12]. There is no single point of failure because everything is documented. But when it comes to digital money security, blockchain technology has a few advantages over the alternatives. Participation and data access might differ amongst blockchain networks. Public and private networks are the two most common varieties.

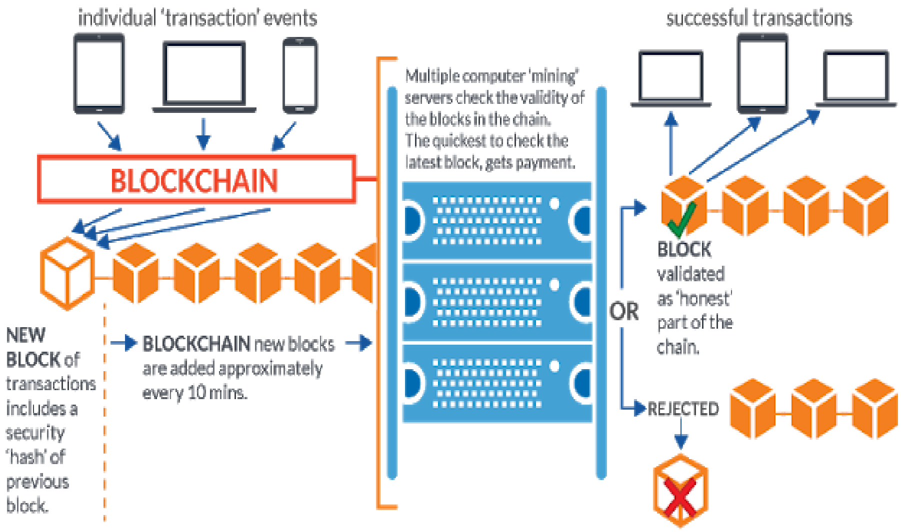

Fig. 1. Dynamic Cryptography Based Security in Blockchain

A common feature of public blockchain networks is the ability for anybody to join while yet maintaining their anonymity. Using computers linked to the internet, a public blockchain can verify transactions [13] and establish consensus. Public blockchains, like as Bitcoin, use "bitcoin mining" as a means of bringing parties together. The "miners" on the bitcoin network work cooperatively to solve a complex cryptographic challenge in order to validate a transaction. Other than public keys, this type of network has few means of identity and access control.

In private blockchains, where only well-known companies are allowed to participate, identity is employed to authenticate membership and access credentials [14, 15, 16]. The organisations' combined efforts establish an exclusive "business network." Consensus on a private blockchain in a permissioned network is achieved by "selective endorsement."

Before choosing a network for any blockchain application, think about the requirements of your firm. It is better for requirements to use permissioned and private networks, as they may be more tightly controlled [17, 18]. Decentralization and dispersion can take place more easily in permissionless networks. Table 1 is the key tools for secured blockchain

Table 1. Key Tools for Secured Blockchain Environment

URL	Framework / Tool
https://etherlime.readthedocs.io/en/latest/	Etherlime

https://ethfiddle.com/	EthFiddle
https://embark.status.im/	Embark
http://populus.readthedocs.io/en/latest/	Populus
http://remix.ethereum.org/	Remix IDE
https://truffleframework.com/	Truffle
https://geth.ethereum.org/ https://github.com/ethereum/go-ethereum/wiki/geth	Go Ethereum / Geth
https://consensys.net/diligence/mythril.html	MyThril
https://github.com/cryppadotta/dotta-license/tree/master/dot-abi-cli	Dot-Abi-cli
https://github.com/ethereum/pyethereum	PyEthereum
https://nethereum.com/	Nethereum
https://github.com/consensys/cava	Cava
http://www.liquidity-lang.org/	Liquidity
https://infura.io/	Infura
https://lamden.io/	Lamden
http://solidity.readthedocs.io/en/v0.4.24/	Solidity
https://coq.inria.fr/	Coq

3 Key Objective and Related Aspects

The unique verification process of Blockchain is one of its outstanding features. By removing the need for human verification, Blockchain promises to increase accuracy. Third-party verification costs have also been reduced as a result of the use of the blockchain technology [19]. Because of its decentralised design, hackers and attackers have had a tough time tampering with data. Security, privacy, and efficiency are just a few of the benefits of using Blockchain to do business. Since it is a transparent technology, it provides complete transparency to its users. As a result, inhabitants of countries with unsafe or undeveloped governments can use Blockchain as a financial option as well as a way to protect their personal information [20]. It's still early days for blockchain, but there are countless chances for professionals to study and grow their careers in this industry, including cryptography in blockchain for sure.

There are blocks of various data items and documents in the blockchain network. It is impossible to alter a block after it has been added to the blockchain. In this context, "immutable" refers to the fact that it cannot be changed or tampered with. Consequently, it creates a secure chain of blocks that eliminates the risk of data manipulation or leaking [21-24].

The Genesis Block is the initial entry in the chain of the network and is where the blockchain begins transactions. It gets more difficult to decipher the prior states because of the various encryptions that have been added to each new block that is inserted. Table 2 show the literature review.

Table 2. Literature Review

Authors	Key Work
G. Zyskind, A.S. Pentland [25]	Using Secured Cryptography based blockchain, data exchanges between users and apps may be protected and undamaged. Network nodes reward trustworthy nodes for their degree of trust rather than proof-of-work.

Authors	Key Work
B. Benshoof, A. Rosen, A.G. Bourgeois, R .W. Harrison [26]	Cryptography Hash with Dynamic Security and "D3NS" with blockchain-based solution for securing DNS. New DNS proposed that is backwards compatible.
A. Ouaddah, A. Abou Elkalam, A. Ait Ouahman [27]	PoC Based Implementation
M. Ali, *et al. [28]*	Dynamic Cryptography based application., "BlockStack" is a test project for immutable data naming and storage. Recognizing that the Namecoin blockchain does not provide the same level of security and trustworthiness as the Bitcoin blockchain.
K. Christidis, M. Devetsikiotis [29]	IoT devices with implementation patterns with the use of blockchain technology are examined in detail.

4 Methodology

- Using Python based Programming Platform for Cryptography in Blockchain Development
- Development, Generation and Deployment of Dynamic Hash
- Using Dynamic Hash in Blockchain Environment

Python is the programming language of choice for high-performance computations in practically every field. The tools and frameworks provided by Python may be used to construct blockchain applications, including those that are decentralized.

```
$ pip install <packagename>

MyDrive:\PythonInstallationDirectory>python -m pip in-
stall <packagename>
```

The incorporation of dynamic cryptographic and encryption into the blockchain technology is critical and heavily relied upon. Installing the hashlib library is as simple as following the instructions found in the previous paragraph.

To ensure that all transactions and records are safe, a secure blockchain generates hash values for each transaction and record. It is possible to produce the dynamic hash value needed to build a blockchain from a collection of individual transactions by running the following script, blockchainhash.py.

```
ImportLibrary datetime as date
ImportLibrary hashlibrary as hashlibraryer
ClassDeclaration BlockchainId:
```

```
   def __init__(self, myindex, ts, myBlock-
chainIdchain, backhashlibrary):
   self.myindex = myindex
   self.ts = ts
   self.myBlockchainIdchain = myBlockchainId-
chain
   self.backhashlibrary = backhashlibrary
   self.hashlibrary = self.hashlibraryop()

   def hashlibraryop(self):
   shahashlibrary = hashlibraryer.sha256()
   shahashlibrary.update(str(self.myindex) +
str(self.ts) + str(self.myBlockchainIdchain) +
str(self.backhashlibrary))
   return shahashlibrary.hexdigest()
  def initializeBlockchainId():
   return BlockchainId(0, date.datetime.now(),
"InitializeBlockchainId BlockchainId", "0")
  def next_BlockchainId(last_BlockchainId):
   this_myindex = last_BlockchainId.myindex + 1
   this_ts = date.datetime.now()
   this_myBlockchainIdchain = "BlockchainId" +
str(this_myindex)
   this_hashlibrary =
last_BlockchainId.hashlibrary
   return BlockchainId(this_myindex, this_ts,
this_myBlockchainIdchain, this_hashlibrary)
  BlockchainIdchain = [initializeBlockchainId()]
  back_BlockchainId = BlockchainIdchain[0]
  maxBlockchainIds = 20
  for i in range(0, maxBlockchainIds):
   BlockchainId_to_add =
next_BlockchainId(back_BlockchainId)
   BlockchainIdchain.append(BlockchainId_to_add)
   back_BlockchainId = BlockchainId_to_add
   print "BlockchainId #{} inserted in Block-
chainId-
chain".format(BlockchainId_to_add.myindex)
   print "Hashlibrary Value:
{}\n".format(BlockchainId_to_add.hashlibrary)
```

5 Result

The result of executing code is a different hash value and a better degree of security employing cryptography techniques. Attempts to hack or sniff the transaction will be nearly impossible if these hash values are used. Figure 2 shows the result of secured hash generation in blockchain.

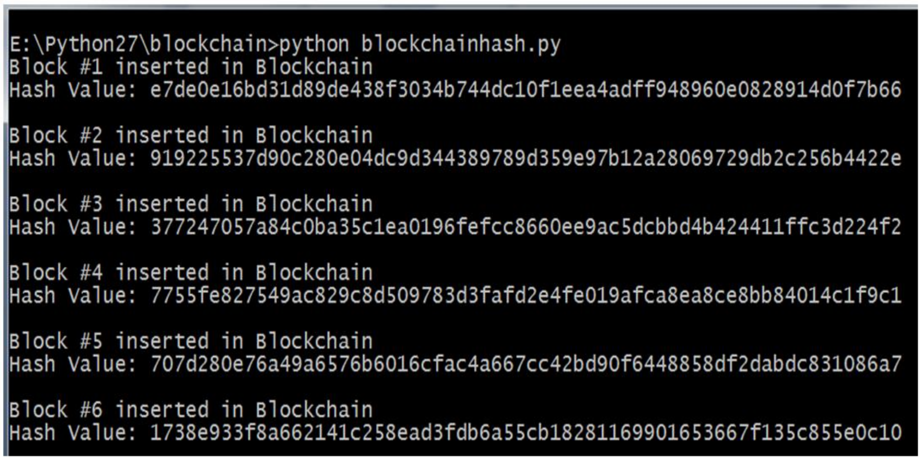

Fig. 2. Secured Hash Generation in Blockchain

6 Deployment of Network Based Distributed Blockchains

Block-based hash functions, like those in preceding examples, are implemented on a standalone system. The real blockchain must be dispersed in order for various users to start their own transactions and blocks. Python has a variety of frameworks for distributed and web-based solutions.

Thus, the digital currency or transaction is carried out securely. B's records need to reflect the value of a file or digital currency sent by A, for example, if A's records need to be wiped and mirrored in B's records. Traditionally, the Bank has acted as a go-between in this transaction. On the blockchain, transactions are validated in real time by specialised algorithms, cutting out the middlemen. Currency will depreciate regardless of the kind of currency if the sender does not erase the transaction from their account. Fig. 3. Shows blockchain initialization.

```
  CryptoMiner_address = "***************************"
  mySecuredBlockchain = []
  mySecuredBlock-
chain.append(Creation_genesis_SecuredBlock())
  ThisClass_Secured_Nodes_SecuredTransactions = []
  peer_Secured_Nodes = []
  SecuredMining = True
```

```
  @Secured_Node.route('/mySecuredBlockchain', meth-
ods=['POST'])
  def SecuredTransaction():
   new_mySecuredBlockchain = request.get_json()
   This-
Class_Secured_Nodes_SecuredTransactions.append(new_mySecu
redBlockchain)
   Display "New SecuredTransaction"
   Display "Sender:
{}".format(new_mySecuredBlockchain['from'].encode('ascii'
,'replace'))
   Display "Receiver:
{}".format(new_mySecuredBlockchain['to'].encode('ascii','
replace'))
   Display "Amount:
{}\n".format(new_mySecuredBlockchain['amount'])
   return "SecuredTransaction Successful\n"
  @Secured_Node.route('/SecuredBlocks', methods=['GET'])
  def get_SecuredBlocks():
   chain_to_send = mySecuredBlockchain
   for i in range(len(chain_to_send)):
   SecuredBlock = chain_to_send[i]
   SecuredBlock_idx = str(SecuredBlock.idx)
   SecuredBlock_timestamp = str(SecuredBlock.timestamp)
   SecuredBlock_data = str(SecuredBlock.data)
   SecuredBlock_hash = SecuredBlock.hash
   chain_to_send[i] = {
   "idx": SecuredBlock_idx,
   "timestamp": SecuredBlock_timestamp,
   "data": SecuredBlock_data,
   "hash": SecuredBlock_hash
   }
```

Fig. 3. Blockchain Initialization

```
$ curl "http://localhost:5000/blockchain" -d
"{\"from\":\"ss\",\"to\":\"fsd\", \"amount\":3}" -H "Con-
tent-Type:application/json"
```

Fig. 4. Curl based Cryptography Hash Initialization

Proof-of-work (PoW) is a critical algorithm in blockchain programming. It is used to verify and confirm transactions in order to add new blocks to the blockchain. The key consensus algorithm for verifying and authenticating transactions is referred to as such. In a blockchain network, there are a variety of miners that work together to verify and finalise transactions. The miners are compensated with digital crypto-currencies as compensation for their successful validations. Figure 4-6 shows the execution process.

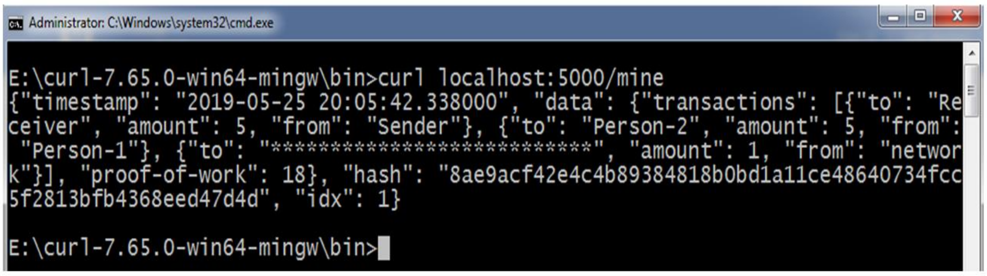

Fig. 5. Secured Transactions

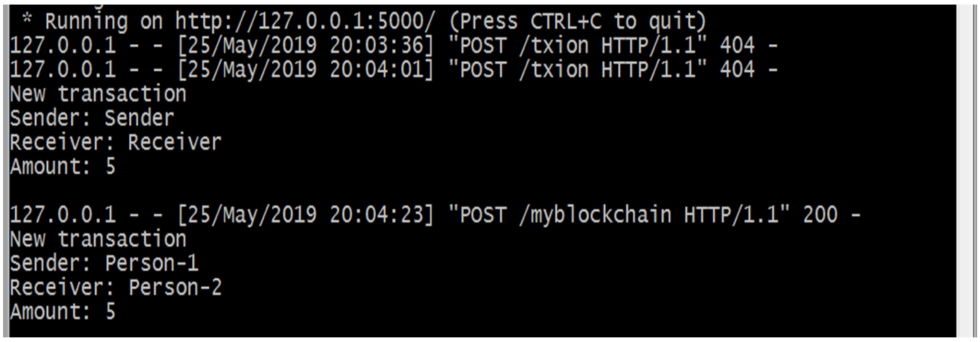

Fig. 6. Secured Transaction with Cryptography in Users

With the code execution and overall implementation described, there will be no effort at hacking because all data and transactions can be inspected to ensure complete transparency. It is possible to record and commit the integrity of transactions by utilising Proof of Work (PoW).

7 Conclusion

Currently, governments and corporations alike are working to protect their applications by implementing blockchain technology. Secure Proof of Work (PoW) algorithms must be linked to these integrations in order to ensure implementation privacy and integrity. Blockchain technology may be utilised by researchers and forensic scientists to accurately anticipate the identities of certain individuals, which can be employed in criminal forensic and law enforcement settings. Software (Electrum, Bitcoin core) and perhaps a specific hardware device (e.g. Ledger) can be used to store transaction data and the user's private and public keys (e.g. private/public key pair). It's critical to understand that these wallets do not hold any actual money (e.g. Bitcoin, Ethereum). There is nothing more to these wallets than a location to store one's private keys and transaction balance. A blockchain wallet is also required to conduct transactions with these other users. To put it another way, the blockchain holds all of the true information/data and cash in blocks, not a wallet. It's akin to a digital signature, which serves as a kind of identification for both the recipient and the whole blockchain network. A particular method must be used to combine your data and your cryptographic signature to establish a unique digital signature each time you begin a transaction with another node. As a result, you may rest assured that both your node and also the data it transmits are genuine.

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Article submitted 3 July 2022. Published as resubmitted by the authors 1 August 2022.

Jordan-Lie Inner Ideals of the Orthogonal Simple Lie Algebras

https://doi.org/10.31185/wjcm.Vol1.Iss2.39

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Abstract—Let A be an associative algebra over a field $\mathbb{F}$ of any characteristic with involution $*$ and let $K = skew(A) = \{a \in A | a^* = -a\}$ be its corresponding sub-algebra under the Lie product $[a, b] = ab - ba$ for all $a, b \in A$. If $A = EndV$ for some finite dimensional vector space over $\mathbb{F}$ and $*$ is an adjoint involution with a symmetric non-alternating bilinear form on V, then $*$ is said to be orthogonal. In this paper, Jordan-Lie inner ideals of the orthogonal Lie algebras were defined, considered, studied, and classified. Some examples and results were provided. It is proved that every Jordan-Lie inner ideals of the orthogonal Lie algebras is either $B = eKe^*$ or B is a type one point space.

Keywords—paper publishing, journals, styles, how-to

1 Introduction

Let A be a finite dimensional associative algebra over a field $\mathbb{F}$. Recall that A becomes a Lie algebra $A^{(-)}$ under the Lie bracket defined by $[x, y] = xy - yx$ for all $x, y \in A$. Suppose that A has an involution $*$. Recall that an involution is a linear transformation $*$ of an algebra A satisfying $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. We denote by $K = skew(A) = \{a^* = -a | a \in A\}$ to be the vector space of the skew symmetric elements of A. Recall that K is a Lie algebra with the Lie bracket defined by $[x, y] = xy - yx$ for all $x, y \in K$. If the characteristic of $\mathbb{F}$ is non-equal 2, then K can be represented in the form:

$$\boldsymbol{K} = \boldsymbol{skew}(\boldsymbol{A}, *) = \{\boldsymbol{a} - \boldsymbol{a}^* | \boldsymbol{a} \in \boldsymbol{A}\}. \quad (1.1)$$

Benkart was the first to introduce an inner ideal of a Lie algebra. She defined it as a subspace $\boldsymbol{B}$ of a Lie $\boldsymbol{L}$ such that the space $[\boldsymbol{B}, [\boldsymbol{B}, \boldsymbol{L}]]$ is a subset of $\boldsymbol{B}$ [4]. She highlighted the relationship between inner ideals and an $\boldsymbol{ad}$-nilpotent elements [3]. Recall that an adjoint map $\boldsymbol{ad}: \boldsymbol{L} \to \mathfrak{gl}(\boldsymbol{L})$ is a representation from a Lie $\boldsymbol{L}$ into its general linear algebra defined by $\boldsymbol{ad}(\boldsymbol{\ell}) = \boldsymbol{ad}_{\boldsymbol{\ell}}$, where $\boldsymbol{ad}_{\boldsymbol{\ell}}: \boldsymbol{L} \to \boldsymbol{L}$ is a linear transformation defined by $\boldsymbol{ad}_{\boldsymbol{\ell}}(\boldsymbol{x}) = [\boldsymbol{\ell}, \boldsymbol{x}]$ for all $\boldsymbol{x} \in \boldsymbol{L}$. By restricting $\boldsymbol{ad}$-nilpotent elements, one can classify non-classical from classical simple Lie algebras over algebraically closed fields of characteristic $\neq \mathbf{2}, \mathbf{3}$. Therefore, inner ideals play a role in classifying these algebras. Commutative inner ideals have proved to be a useful tool for classifying both finite and infinite-dimensional simple Lie algebras. It is proved in [9] that inner ideals play a role similar to one-sided ideal in associative algebras and can be used to construct Artinian structure theory for Lie algebras. Inner ideals is an essential tool in the classification of Lie algebras. (see [8] and [9]). Inner ideals of classical type Lie sub-algebras of associative(simple) rings were studied by Benkart and Fernandez Lopez (see[6]) . Baranov and Shlaka [2] in 2019 classified Jordan-Lie inner ideals of the Lie sub-algebras of finite dimensional associative algebras. An inner ideal $\boldsymbol{B}$ of $\boldsymbol{A}^{(\boldsymbol{k})}$ or $\boldsymbol{K}^{(\boldsymbol{k})}$ is said to be Jordan-Lie if $\boldsymbol{B}^{\mathbf{2}} = \mathbf{0}$. In recent paper, Shlaka and Mousa [11], studied Jordan-Lie inner ideals $\boldsymbol{A}^{(\boldsymbol{k})}$ in the case when $\boldsymbol{A}$ is simple over an algebraically closed fields of positive characteristic. Jordan-Lie inner ideals of the Lie algebras $\boldsymbol{K}^{(\boldsymbol{k})}$ in the case when $\boldsymbol{A}$ is simple with the symplectic involution over an algebraically closed fields of positive characteristic were also been studied by Kareem and Shlaka in [7].

In this paper, we study inner ideals of the orthogonal Lie algebras. We start with some preliminaries in section $\mathbf{2}$. Section $\mathbf{3}$ is devoted to proof some results about Jordan-Lie inner ideals of the orthogonal Lie algebras and point space.

2 Preliminaries

Throughout this paper, $\mathbb{F}$ is a field (algebraically closed), $p \geq 0$ is the characteristic of $\mathbb{F}$, V is a vector space (finite dimensional over $\mathbb{F}$), $\mathrm{E}nd(V)$ is the endomorphism algebra, $\mathfrak{so}(V)$ is the orthogonal Lie algebra, A is an associative algebra (finite dimensional over $\mathbb{F}$) with an involution $*$, $K = skew(A,*)$ is the Lie subalgebra of A defined as (1.1), L is a Lie algebra (finite dimensional over $\mathbb{F}$), $\mathcal{M}_n(\mathbb{F})$ is the matrix algebra consisting of all $n \times n$-matrices and $\mathfrak{so}_n(\mathbb{F})$ is the orthogonal Lie algebra of matrix .

Recall that an *involution* $*$ of A is a linear transformation of A such that $(a^*)^* = a$ and $(ab)^* = b^* a^*$ for any $a, b \in A$ [10]. Note that $*$ does not required to be $\mathbb{F}$-linear. On the other hand, it is obvious that $*$ maps the center Z into it self. Since the restriction of $*$ over $\mathbb{F}$ is an automorphism of order less than or equal to 2, it maps every sub-field of Z into itself. Therefore $\mathbb{F}^* = \mathbb{F}$. Here we have two possibilities which are either $*$ is $\mathbb{F}$-linear or not. Thus, we have the following definition.

2.1 **Definition** [13, 7.2] An involution is said to be of the first kind in case that $*$ is $\mathbb{F}$-linear, that is the restriction of $*$ relative to $\mathbb{F}$ is the identity. Otherwise, it is called of the second kind.

2.2 **Remark** In this paper, we consider involution of the first kind only.

2.3 **Definition** Let $\boldsymbol{B}$ be a subspace of **L**. Then **B** is said to be

1. [4] An inner ideal if $\big[B,[B,L]\big] \subseteq B$.
2. [4] A commutative inner ideal if B is an inner ideal such that $[B,B] = 0$.
3. [2] A Jordan-Lie inner ideal (or simply, J-Lie) if $L = skew(A)$ and B is an inner ideal such that $B^2 = 0$.

2.4 **Example** *Consider the associative algebra $\boldsymbol{A} = \boldsymbol{\mathcal{M}_n}(\mathbb{F})$. Then $\{\boldsymbol{e_{ij}} | \mathbf{1} \le \boldsymbol{i}, \boldsymbol{j} \le \boldsymbol{n}\}$ form a basis of $\boldsymbol{A}$ consisting of matrix units, where $\boldsymbol{e_{ij}}$ is the $\boldsymbol{n} \times \boldsymbol{n}$-matrix with the entry $\mathbf{1}$ in the $\boldsymbol{ij}$th position and zero elsewhere. Thus, the Lie algebra*

$K = skew(A) = \mathfrak{so}_{2n}(\mathbb{F})$ has the following basis $\{a_{ij}, b_{ij}, c_{ij} | 1 \le i, j \le n\}$, where $a_{ij} = (e_{ij} - e_{n+j,n+i})$, $b_{ij} = (e_{i,n+j} - e_{j,n+i})$ and $c_{ij} = (e_{n+i,j} - e_{n+j,i})$.

Then $B = \mathbb{F}a_{12}$ is J-Lie of $skew(A,*)$. Indeed, for any $x, y \in B$, we have $x = \alpha a_{12} = \alpha(e_{12} - e_{n+2,n+1})$, $y = \beta a_{12} = \beta(e_{12} - e_{n+2,n+1})$. Since

$$x.y = \alpha(e_{12} - e_{n+2,n+1}).\beta(e_{12} - e_{n+2,n+1}) = 0,$$

$B^2 = 0$. It remain to show that $[x,[y,\ell]] \in B$ for each $\ell \in K$.

Let $\ell = \sum_{i,j=1}^{n} \zeta_{ij} a_{ij} + \sum_{i,j=1}^{n} \eta_{ij} b_{i,n+j} + \sum_{i,j=1}^{n} \gamma_{ij} c_{ij} \in K$. Then

$$x\ell y = \alpha(e_{12} - e_{n+2,n+1})(\sum_{i,j=1}^{n} \zeta_{ij} a_{ij} + \sum_{i,j=1}^{n} \eta_{ij} b_{ij} + \sum_{i,j=1}^{n} \gamma_{ij} c_{ij})y$$

$$= \alpha \sum_{j=1}^{n} (\zeta_{2j} e_{1j} + \eta_{2j} e_{1,n+j} - \eta_{j2} e_{1,n+j} + \zeta_{j1} e_{n+2,n+j} - \gamma_{1j} e_{n+2,j} + \gamma_{j1} e_{n+2,j})y$$

$$= \alpha\beta(\zeta_{21} e_{12} - \eta_{22} e_{1,n+1} - \eta_{22} e_{1,n+1} - \zeta_{21} e_{n+2,n+1} - \gamma_{11} e_{n+2,2} + \gamma_{11} e_{n+2,2})$$

$$= \alpha\beta\zeta_{21}(e_{12} - e_{n+2,n+1}) = \alpha\beta\zeta_{21} a_{12} \in \mathbb{F}a_{12} = B.$$

and

$$y\ell x = \beta(e_{12} - e_{n+2,n+1})(\sum_{i,j=1}^{n} \zeta_{ij} a_{ij} + \sum_{i,j=1}^{n} \eta_{ij} b_{ij} + \sum_{i,j=1}^{n} \gamma_{ij} c_{ij})x$$

$$= \beta \sum_{j=1}^{n} (\zeta_{2j} e_{1j} + \eta_{2j} e_{1,n+j} - \eta_{j2} e_{1,n+j} + \zeta_{j1} e_{n+2,n+j} - \gamma_{1j} e_{n+2,j} + \gamma_{j1} e_{n+2,j}) x$$

$$= \alpha\beta(\zeta_{21} e_{12} - \eta_{22} e_{1,n+1} - \eta_{22} e_{1,n+1} - \zeta_{21} e_{n+2,n+1} - \gamma_{11} e_{n+2,2} + \gamma_{11} e_{n+2,2})$$

$$= \alpha\beta\zeta_{21}(e_{12} - e_{n+2,n+1}) = \alpha\beta\zeta_{21} a_{12} \in \mathbb{F}a_{12} = B.$$

Therefore, $[x,[y,\ell]] = xy\ell - x\ell y - y\ell x + \ell yx = -x\ell y - y\ell x \in B$, as required.

2.5 Definition [5] A subspace **P** of **L** is said to be point space if $[\mathbf{P},\mathbf{P}] = \mathbf{0}$ and $\mathbf{ad}_{\mathbf{x}}^{\mathbf{2}}(\mathbf{L}) = \mathbb{F}\mathbf{x}$ for every non zero element $\mathbf{x} \in \mathbf{P}$.

Example 2.6 *Let* $K = \mathfrak{so}_{2n+1}(\mathbb{F})$, *If* $n = 1$, *then*

$$K = \mathfrak{so}_3(\mathbb{F}) = span\{\begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_3 & 0 \\ -\alpha_1 & 0 & -\alpha_3 \end{pmatrix} | \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}\}$$

has a basis are

$$\{b_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\}$$

Then , we need to show that b_1, is a point space. For $x \in \mathbb{F}b_1$ we have

$x = \begin{pmatrix} 0 & \zeta & 0 \\ 0 & 0 & 0 \\ -\zeta & 0 & 0 \end{pmatrix}$ for some $\zeta \in \mathbb{F}$. Let $\ell = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_3 & 0 \\ -\alpha_1 & 0 & -\alpha_3 \end{pmatrix} \in \mathfrak{so}_3(\mathbb{F})$.

Then, $\mathrm{a}d_x^2(L) = [x,[x,\ell]]$

$$= [x, \begin{pmatrix} -\zeta\alpha_2 & \zeta\alpha_3 & 0 \\ 0 & 0 & 0 \\ 0 & -\zeta\alpha_1 & -\zeta\alpha_2 \end{pmatrix} - \begin{pmatrix} -\zeta\alpha_2 & 0 & 0 \\ 0 & -\zeta\alpha_2 & 0 \\ \zeta\alpha_3 & -\zeta\alpha_1 & 0 \end{pmatrix}]$$

$$= [\begin{pmatrix} 0 & \zeta & 0 \\ 0 & 0 & 0 \\ -\zeta & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta\alpha_3 & 0 \\ 0 & \zeta\alpha_2 & 0 \\ -\zeta\alpha_3 & 0 & -\zeta\alpha_2 \end{pmatrix}]$$

$$= \begin{pmatrix} 0 & \zeta^2\alpha_2 & 0 \\ 0 & 0 & 0 \\ 0 & -\zeta^2\alpha_3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \zeta^2\alpha_2 & -\zeta^2\alpha_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \zeta^2\alpha_2 & 0 \\ 0 & 0 & 0 \\ -\zeta^2\alpha_2 & 0 & 0 \end{pmatrix} \in \mathbb{F}b_1$$

Therefore, $\mathbb{F}b_1$ and also $\mathbb{F}b_2$ is a point space. while $\mathbb{F}b_2$ is not point space.
We will need the following lemma. For the proof see [12].
Lemma2.7 *[12] Let B be an I-ideal of L. If $B^2 = 0$, then*
(1) $b_1\ell b_2 + b_2\ell b_1 \in B$ for all $b_1, b_2 \in B$ and $\ell \in L$.
(2) $b\ell b \in B$ for all $b \in B$ and $\ell \in L$.

2.6 **Definition** [10] Let $\boldsymbol{\psi}: \mathbf{V} \times \mathbf{V} \to \mathbb{F}$ be a nondegenerate symmetric bilinear form. For each $\mathbf{x} \in \mathbf{EndV}$ define $\mathbf{x}^*$ by the following property $\boldsymbol{\psi}(\mathbf{x}^*(\mathbf{v}), \mathbf{w}) = \boldsymbol{\psi}(\mathbf{v}, \mathbf{x}(\mathbf{w}))$ for all $\mathbf{v}, \mathbf{w} \in \mathbf{V}$. Then the map $*: \mathbf{EndV} \to \mathbf{EndV}$ is an involution of the algebra $\mathbf{EndV}$, called the adjoint involution with respect to $\boldsymbol{\psi}$.

2.7 **Theorem** [10, Ch.1, introduction] The map $\boldsymbol{\psi} \mapsto *$ induced one to one correspondence between equivalence classes of nondegenerate bilinear forms on $\boldsymbol{V}$ modulo multiplication by a factor in $\mathbb{F}^\times$ and involution (of first kind) on $\boldsymbol{EndV}$.

2.8 **Definition** [10] Let $*$ be an involution of $\mathbf{EndV}$. We say that $*$ is orthogonal if it is adjoint to a symmetric non-alternating bilinear form on $\mathbf{V}$.

2.9 **Definition** *[3] Let $\boldsymbol{A}$ be an associative algebra with involution $*$ over a field $\mathbb{F}$and let $\boldsymbol{a} \in \boldsymbol{A}$. Then we define the trace of $\boldsymbol{a}$ by $\boldsymbol{\tau}(\boldsymbol{a}) = \boldsymbol{a} - \boldsymbol{a}^*$.*

3 Jordan Lie inner ideal of the orthogonal Lie algebras

3.1 **Theorem** Suppose that $\mathbf{A}$ is simple with involution and $\mathbf{p} \neq \mathbf{2}$. Let $\mathbf{x} \in \mathbf{skew}(\mathbf{A},*)$. Then $\mathbf{x} = \mathbf{xyx}$ for some $\mathbf{y} \in \mathbf{skew}(\mathbf{A},*)$.

Proof. We have $x^* = -x$. Since A is V-Neumann algebra, $x = xax$ for some $a \in A$. Put $y = \frac{1}{2}(a - a^*) \in skew(A,*)$. Then

$xyx = \frac{1}{2}x(a - a^*)x = \frac{1}{2}(xax - xa^*x) = \frac{1}{2}(x - (xax)^*) = \frac{1}{2}(x - x^*) = \frac{1}{2}(2x) = x$. ■

3.2 Lemma Let $\mathbf{eKe^* \subseteq B}$ be a subspace of $\mathbf{K = skew(A,*)}$ such that $\mathbf{e \in BK}$ and $\mathbf{e^* \in KB}$. If $\mathbf{e'}$ be an idempotent in **A** such that $\mathbf{ee' = e'e = 0}$, then $\mathbf{e'Be'^* \subseteq B}$.

Proof. If $e'Be'^* = 0$. Then $e'Be'^* \subseteq B$. Suppose now that $e'Be'^* \neq 0$. Then $\exists a \in B$ such that $e'ae'^* \neq 0$.

$$e'ae'^* = (1-e)a(1-e^*) = a - (ea + ae^*) + eae^*$$

As $e \in BK$, $\exists b_1 \in B$ and $k_1 \in K$ such that $e = b_1k_1$. This implies that

$$e^* = (b_1k_1)^* = {k_1}^* b^* = k_1b_1$$

We have $a \in B$ and $eae^* \in eKe^* \subseteq B$. By Lemma 2.7,

$$ea + ae^* = b_1k_1a + ak_1b_1 \in B$$

Therefore, $e'ae'^* \in B$, as required.■

Recall that A is simple, so A can be identified with $End(V)$ for some vector space V. We have the following proposition.

3.3 Proposition Let $\boldsymbol{\psi}: \mathbf{V \times V \to \mathbb{F}}$ be a non-singular form and let $*$ be an adjoint involution of $\mathbf{A = End(V)}$. Let $\mathbf{e, e'}$ be idempotent in **A** such that $\mathbf{ee' = e'e =}$ $\mathbf{0}$. Suppose that $\mathbf{eKe^* \neq 0}$. Then the following hold

For each $k \in K$ such that $eke^* \neq 0$, we have
(1) $c = k + e'ke'^* \neq 0$.
(2) $e'Ke^* = 0$.
(3) $eKe'^* = 0$.

Proof. (1) Let $v \in V$ such that $\psi(v, eke^*(v)) \neq 0$. Such v exists because ψ is non-singular. We need to show that $\psi(e^*(v), ce^*(v) \neq 0$. Since $ee' = 0$,

$$\psi(e^*(v), ce^*(v)) = \psi(v, ece^*(v))$$

$$= \psi(v, e(k + e'ke'^*)e^*(v))$$

$$= \psi(v, eke^*(v)) + \psi(v, ee'ke'^*e^*(v))$$

$$= \psi(v, eke^*(v)) \neq 0.$$

(2) Let $w \in e'Ke^*$. Then there is $k \in K$ such that $w = e'ke^*$. For each $v \in V$ we have

$$\psi(e^*(v), we^*(v)) = \psi(v, ewe^*(v)) = \psi(v, ee'ke^*e^*(v)) = \psi(v, 0) = 0,$$

so $w = e'Ke^* = 0$.

(3) Let $h \in eKe'^*$. Then there is $k \in K$ such that $h = eke'^*$. For each $v \in V$ we have

$\psi(e^*(v), he^*(v)) = \psi(v, ehe^*(v)) = \psi(v, e(eke'^*)e^*(v)) = \psi(v, 0) = 0.$

Therefore, $h = eke'^* = 0$. ■
The idea of the following lemma comes from McCrimmon's paper [3].

3.4 Lemma Let **A** be an associative algebra with involution $*$ over a field $\mathbb{F}$. Suppose that $\mathbf{L} = \mathbf{skew}(\mathbf{A}, *)$. Then the trace $\boldsymbol{\tau}$ that defined above by $\boldsymbol{\tau}(\mathbf{a}) = \mathbf{a} - \mathbf{a}^*$ has the following properties:

(1) τ is linear.

(2) $\tau(x) \in L$ for any $x \in L$.

(3) $x\tau(a)x = \tau(xax)$ For any $a \in A$ and $x \in L$.

(4) $a\tau(b) + \tau(b)a^* = \tau(ab) + \tau(ba^*)$ For any $a, b \in A$.

(5) $\tau(a)x\tau(a) = \tau(axa) - axa^* - a^*xa$ For any $a \in A$ and $x \in L$.

Proof. (1) Suppose that $a, b \in A$ and $\alpha \in \mathbb{F}$. Then

$$\tau(\alpha a) = \alpha a - (\alpha a)^* = \alpha(a - a^*) = \alpha\tau(a);$$

$$\tau(a + b) = (a + b) - (a + b)^* = (a - a^*) + (b - b^*) = \tau(a) + \tau(b).$$

Thus, τ is linear.
(2) Let $a \in A$. Then

$$\left(\tau(a)\right)^* = (a - a *)^* = a^* - a = -(a - a^*) = -\tau(a).$$

Therefore, $\tau(a) \in L$.
(3) Let $a \in A$ and $x \in L$. Then we have

$$x\tau(a)x = x(a - a^*)x = xax - xa^*x = xax - (x^*ax^*)^* = \tau(xax).$$

(4) Let $a, b \in A$. Then

$$a\tau(b) + \tau(b)a^* = a(b - b^*) + (b - b^*)a^* = ab - ab^* + ba^* - b^*a^*$$

$$= (ab - b^*a^*) + (ba^* - ab^*)$$

$$= (ab - (ab)^*) + (ba^* - (ba^*)^*) = \tau(ab) + \tau(ba^*)$$

(5) For any $a \in A$ and $x \in L$ we have

$$\tau(a)x\tau(a) = (a - a^*)x(a - a^*) = axa + a^*xa^* - axa^* - a^*xa$$

$$= (axa - (axa)^*) - axa^* - a^*xa = \tau(axa) - axa^* - a^*xa.$$ ■

3.5 Lemma Suppose that $\mathbf{p} \neq \mathbf{2}, \mathbf{3}$ and $\mathbf{K} = \mathbf{skew}(\mathbf{EndV}, *)$. Then the following hold:

1. If $\psi(Kv, w) = 0$ for some nonzero vectors $v, w \in V$, then $w \in \mathbb{F}v$. Consequently $Kv = v^{\perp}$ for any nonzero vector $v \in V$.
2. If U is a subspace such that $\dim U > 1$, then $KU = V$.
3. A transformation $x \in K$ satisfies $xKx^{*} = 0$ if and only if $rank(x) \leq 1$.

Proof. (1) Suppose that $v, w \in V$ be nonzero vectors such that $\psi(Kv, w) = 0$. For the contrary we assume that $w \notin Kv$. Then we could find a linear transformation $a \in A$ such that $a(w) = 0$ and $\psi(a(v), w) \neq 0$. Note that $a - a^{*} \in K$. Thus,

$$0 \neq \psi(a(v), w) = \psi(a(v), w) - 0 = \psi(a(v), w) - \psi\big(v, a(w)\big)$$

$$= \psi(a(v), w) - \psi(a^{*}(v), w) = \psi\big((a - a^{*})(v), w\big) = 0,$$

a contradiction. Therefore $w \in Kv$. Consequently, for any nonzero vector v we have $v^{\perp} = Kv$.

(2) Suppose that U be a subspace of V such that $\dim U > 1$. Then

$$KU = \sum_{w \in U} Kw = \sum_{w \in U} w^{\perp} = V$$

That is, any $w^{\perp}$has co-dimensional 1. Thus, if $w_1^{\perp} = w_2^{\perp}$. Then $w_1 \in Kw_2$. Hence any two independent vectors $w_i^{\perp}$ will span all V.

(3) If $x^{*}Kx = 0$. Then

$$0 = \psi(x^{*}Kx(v), v) = \psi\big(Kx(v), x(v)\big) \text{ for all } v \in V.$$

This implies $K\big(x(V)\big) \neq V$, so by (2), we get that $\dim\big(x(V)\big) \leq 1$. ■

3.6 Theorem Let $\mathbf{e}, \mathbf{e}', \mathbf{f}$ be an idempotent in $\mathbf{A} = \mathbf{EndV}$ such that $\mathbf{ee}' = \mathbf{e}'\mathbf{e} = \mathbf{0}$ and $\mathbf{e}^{*}\mathbf{e} = \mathbf{0}$. Let $\mathbf{e}^{*}\mathbf{f} = \mathbf{fe}^{*} = \mathbf{0}$ and $\mathbf{e}^{*\prime}\mathbf{f} = \mathbf{fe}^{*\prime} = \mathbf{f}$,If $\mathbf{B} = \mathbf{eKe}'^{*}$ then $\mathbf{B}$ is a $\mathbf{J}$ −Lie.

Proof. Let $w = eke'^{*} \neq 0$, by Theorem 3.1, $w = wz'w$ for some $z' \in K$.
put $z = e'^{*}z'e$. Then

$$wzw = w(e'^{*}z'e)w = eke'^{*}e'^{*}z'eeke'^{*} = eke'^{*}z'eke'^{*} = wz'w$$

Let $f = zw = (e'^{*}z'e)(eke'^{*}) = e'^{*}z'eke'^{*}$. Then

$$e^{*}f = e^{*}e'^{*}z'eke'^{*} = 0$$

and

$$fe^{*} = e'^{*}z'eke'^{*}e^{*} = (1 - e^{*})z'ek(1 - e^{*})e^{*} = 0$$

$$e'^{*}f = e'^{*}e'^{*}z'eke'^{*} = e'^{*}z'eke'^{*} = f$$

Also

$$fe'^* = e'^* z' eke'^* e'^* = f$$

By Lemma 3.5 (3), since $rank(f) = 1$, so $rank(f^*) = 1$. Therefore,

$$f^* K f = (e'^* z' eke'^*)^* K (e'^* z' eke'^*) = e' k^* e^* z'^* e' K e'^* z' eke'^* = 0$$

and

$$f K f^* = e'^* z' eke'^* K e' k^* e^* z'^* e' = 0$$

Moreover, for any $u \in Ker(w)$, $f(u) = zw(u) = 0$

Therefore, $Ker(w) \subseteq Ker(f)$, both have co-dimension. Then

$$Ker(w) = Ker(f) = Ker(w') \quad (3.1)$$

Recall $f = zw$ is idempotent of rank 1. Let $c \in Im(w')$ such that $c \notin Ker(w')$. Then $w' f(c) \neq 0$

(if $w' f(c) = 0$, then either $c \in Ker(f)$ or $c \in Ker(w')$ this is a contradiction)

If $w' f(c) \neq 0$, then $c \in Im(w' f)$. Since $c \in Im(w')$, so $c \in Im(w' f)$

Therefore, $Im(w') \subseteq Im(w' f)$. both have co-dimension, so

$$Im(w') = Im(w' f)$$

Since $Ker(f) = Ker(w')$, so

$$Ker(w' f) = Ker(w')$$

Therefore, $w' f = w'$ for any $w' \in eBe'^*$.

Next, we claim that

$$B \subseteq B' = eKe^* + \tau(\mathrm{ekf}),$$

for any $d \in B$ we have

$$d = ede^* + ede'^* + e'de^* + e'de'^*$$

$$= \mathrm{ede}^* + \mathrm{ede}'^* - (\mathrm{ede}'^*)^*$$

$$= \mathrm{ede}^* + \tau(\mathrm{ede}'^*)$$

$$= \mathrm{ede}^* + \tau(\mathrm{w}')$$

Since $w'(f) = w' \in eBe'^*$, we have that

$$K = eke^* + \tau(w' f) = ede^* + \tau(ede'^* f)$$

As $e'^* f = f$, so

$$K = eke^* + \tau(\mathrm{edf}) \in eKe^* + \tau(eKf)$$

put $B' = eKe^* + \tau(eKf)$. Then

$$(e + f^*) K (e + f^*)^* = (e + f^*) K (e^* + f)$$

$$= eKe^* + eKf + f^*Ke^* + f^*Kf$$

$$= eKe^* + eKf - (eKf)^*$$

$$= eKe^* + \tau(eKf)$$

Let $\mathfrak{g} = e + f^*$, then

$$\mathfrak{g}^2 = (e + e'^* z' e k e'^*)(e + e'^* z' e k e'^*)$$

$$= e + e'^* z' e k e'^* = \mathfrak{g}$$

and $\mathfrak{g}^*\mathfrak{g} = (e^* + f)(e + f^*)$

$$= (e^* + e'^* z' e k e'^*)(e + e' k e^* z'^* e') = 0$$

Now, let $\mathfrak{g}k_1\mathfrak{g}^*, \mathfrak{g}k_2\mathfrak{g}^* \in \mathfrak{g}K\mathfrak{g}$ and $\ell \in K$. Then

$$[\mathfrak{g}k_1\mathfrak{g}^*, [\mathfrak{g}k_2\mathfrak{g}^*, \ell]] = [\mathfrak{g}k_1\mathfrak{g}^*, \mathfrak{g}k_2\mathfrak{g}^*\ell - \ell\mathfrak{g}k_2\mathfrak{g}^*]$$

$$= \mathfrak{g}k_1\mathfrak{g}^*\mathfrak{g}k_2\mathfrak{g}^*\ell - \mathfrak{g}k_2\mathfrak{g}^*\ell\mathfrak{g}k_1\mathfrak{g}^* - \mathfrak{g}k_1\mathfrak{g}^*\ell\mathfrak{g}k_2\mathfrak{g}^* + \ell\mathfrak{g}k_2\mathfrak{g}^*\mathfrak{g}k_1\mathfrak{g}^*$$

$$= -2\mathfrak{g}k_1\mathfrak{g}^*\ell\mathfrak{g}k_2\mathfrak{g}^* = \mathfrak{g}(-2k_1\mathfrak{g}^*\ell\mathfrak{g}k_2)\mathfrak{g}^* \in \mathfrak{g}K\mathfrak{g}^*$$

Therefore, $B' = \mathfrak{g}K\mathfrak{g}^*$ is an I-ideal of $K$and B' is J-Lie of K. as required.■

3.7 Theorem Let $\mathbf{e}, \mathbf{e}', \mathbf{f}$ be an idempotent in **EndV** such that $\mathbf{ee}' = \mathbf{e}'\mathbf{e} = \mathbf{0}$. Let **B** be a $\mathbf{J}$ −Lie of $\mathbf{K} = \mathbf{skew}(\mathbf{A},*)$ such that $\mathbf{bKb} \neq \mathbb{F}\mathbf{b}$ for all $\mathbf{b} \in \mathbf{B}$. Then the following hold

1. $e(V) = eKe^*K(v_0^{\perp})$ for all $v_0 \in V$.
2. $eBe'^*(V) = e(V)$.

Proof. (1) Suppose that $bKb \neq \mathbb{F}b$. We have $B \subseteq B' = (e + f^*)K(e + f^*)^*$
By Lemma 3.5 (1), we have $Kv_0 = v_0^{\perp}$. Thus $eKe^*K(v_0) = eKe^*(v_0^{\perp})$
Suppose that $\mathrm{d}im(e^*(v_0^{\perp})) \geq 1$.
If $\mathrm{d}im(e^*(v_0^{\perp})) > 1$, then by Lemma 3.5 (2),

$$Ke^*(v_0^{\perp}) = V \Longrightarrow eKe^*(v_0^{\perp}) = e(V)$$

Therefore, $e(V) = eKe^*(v_0^{\perp})$.
and if $\mathrm{d}im(e^*(v_0^{\perp})) = 1$, then there exist a non-zero $u_0 \in V$ such that

$$e^*(v_0^{\perp}) = \mathbb{F}u_0. \qquad (3.2)$$

Then

$$eKe^*(v_0^{\perp}) = eK(u_0) = e(u_0^{\perp})$$

for all $u \in u_0^{\perp}$, we have $e(u) \in (v_0^{\perp})^{\perp} = \mathbb{F}v_0$, because

$$e(u) \in e(u_0^{\perp}) = eKe^*(v_0^{\perp}) \subseteq B(v_0^{\perp}) \subseteq K(v_0^{\perp}) = (v_0^{\perp})^{\perp} = \mathbb{F}v_0,$$

so

$$u_0^{\perp} = e'(V) + \mathbb{F}v_0$$

Thus,

$$eKe^*(u_0) = e(u_0^{\perp}) = e(e'(V) + \mathbb{F}v_0) = \mathbb{F}v_0. \quad (3.3)$$

But for any non-zero $r \in u_0^{\perp}$ and $\alpha \in \mathbb{F}$, we have $e^*(r) = \alpha u_0$

$$\alpha e^*(u_0) = e^*(\alpha u_0) = e^*(e^*(r)) = e^*(r) = \alpha u_0.$$

so $e^*(u_0) = u_0$. Thus, for any $y = ey'e^* \in eKe^*$, we can assume that $y(u_0) = 0$

Let $y(V) \subseteq u_0^{\perp}$, by equation (3.3),

$$y(V) = ey'e^*(V) = e(ey'e^*(V)) = e(y(V)) \subseteq e(u_0^{\perp}) = \mathbb{F}v_0$$

By Lemma 3.5 (3), if y has rank 1, then $y^*Ky = yKy = 0$.
By Theorem 3.1, $\exists 0 \neq \ell \in K$ such that $y = y\ell y \in yKy = 0$.
Then, $y \in eKe^* \subseteq B$. Therefore $y \in \mathbb{F}b$, but $eKe^* = bKb$
so $y \in bKb$. Thus, if $bKb \neq \mathbb{F}b$, then $eKe^*K(v_0) = e(V)$, as required.
(2) for any $\ell, \ell' \in K$, we have $e\ell e^* \in eKe^* \subseteq B$.
Let $b'' = -[e\ell e^*, [b', \ell']] \in [B, [B, K]] \subseteq B$

$b' \in B$ is the same b' that satisfies $w = eb'e^{*\prime} \neq 0$. Since

$$b'' = -[e\ell e^*, [b', \ell']] = -[e\ell e^*, b'\ell' - \ell' b']$$

$$= -(e\ell e^* b'\ell' - b'\ell' e\ell e^* - e\ell e^* \ell' b' + \ell' b' e\ell e^*)$$

$$eb''e'^* = -e(e\ell e^* b'\ell' - b'\ell' e\ell e^* - e\ell e^* \ell' b' + \ell' b' e\ell e^*)e'^*$$

$$= -ee\ell e^* b'\ell' e'^* + eb'\ell' e\ell e^* e'^* + ee\ell e^* \ell' b' e'^* - e\ell' b' e\ell e^* e'^*$$

$$= -e\ell e^* b'\ell' e'^* + e\ell e^* \ell' b' e'^*$$

and $e\ell e^* b'\ell' e'^* = bx\ell xbb'\ell' e'^* = 0$. As $(bb' = 0)$

$$eb''e'^* = e\ell e^* \ell' b' e'^* = e\ell e^* \ell'(e + e')b'e'^*$$

$$= e\ell e^* \ell' eb'e'^* + e\ell e^* \ell'(e'b'e'^*)$$

By using equation , $(e'b'e'^* = 0)$, we have $eb''e'^* = e\ell e^* \ell'(eb'e'^*)$.
Since $w = eb'e'^*$, so $eb''e'^* = e\ell e^* \ell' w$ for any $\ell, \ell' \in K$.
Let $v \in V$. Then $eb''e'^*(v) = e\ell e'^* \ell' w(v) = e\ell e'^* \ell'(v_0)$
Since $bKb \neq \mathbb{F}b$, so we must have

$$eBe'^*(V) = eKe^*K(v_0)$$

Since $eKe^*K(v_0) = e(V)$, we get that

$$eBe'^*(V) = e(V)$$

as required. ∎

3.8 **Theorem** Let $\mathbf{e}, \mathbf{f}$ be an idempotent in $\mathbf{A} = \mathbf{EndV}$ and let $\mathbf{B}$ be a $\mathbf{J}$ −Lie of $\mathbf{K} = \mathbf{skew}(\mathbf{A},*)$. Suppose that $\mathbf{bKb} = \mathbb{F}\mathbf{b}$ for all $\mathbf{b} \in \mathbf{B}$.Then $\mathbf{B}$ is a type one point space.

Proof. Suppose that $bKb = \mathbb{F}b$. we are going to prove that B is a type one point space

Recall that $B \subseteq B' = eKe^* + \tau(eKf)$, so

$$B' = eKe^* + \tau(eKf) = bxKxb + \tau(eKf)$$

$$= bKb + \tau(eKf) \qquad (3.4)$$

Since $bKb = \mathbb{F}b$, so

$$B' = \mathbb{F}b + \tau(eKf)$$

for any $c \in B'$, there exist $\lambda \in \mathbb{F}$ and $\ell \in K$ such that

$$c = \lambda b + \tau(e\ell f)$$

Then $\forall y \in K$, we have

$$cyc = (\lambda b + \tau(e\ell f))y(\lambda b + \tau(e\ell f))$$

$$= \lambda^2 byb + \lambda by\tau(e\ell f) + \lambda\tau(e\ell f)yb + \tau(e\ell f)y\tau(e\ell f)$$

$$= \lambda^2 byb + \lambda(by)\tau(e\ell f) + \lambda\tau(e\ell f)(by)^* + \tau(e\ell f)y\tau(e\ell f)$$

By Lemma 3.4 (3),

$$cyc = \lambda^2 byb + \lambda\tau(bye\ell f) + \lambda\tau(e\ell f(by)^*)$$

$$+\tau(e\ell fye\ell f) - e\ell fy(e\ell f)^* - (e\ell f)^* ye\ell f$$

$$\lambda^2 byb + \lambda\tau(bye\ell f) + \lambda\tau(e\ell fyb) + \tau(e\ell fye\ell f) - e\ell fyf^*\ell^*e^* - f^*\ell^*e^*ye\ell f$$

Since $fKf^* = f^*Kf = 0$.

$$cyc = \lambda^2 byb + \lambda\tau(bye\ell f) + \lambda\tau(e\ell fyb) + \tau(e\ell fye\ell f) \qquad (3.5)$$

we need to calculate each term. Since $bKb = \mathbb{F}b$, so

$$byb = \alpha b \qquad (3.6)$$

$$\tau(bye\ell f) = \tau(bybx\ell f) = \tau(\alpha bx\ell f) = \tau(\alpha(e\ell f))$$

$$= \alpha\tau(e\ell f) \quad (3.7)$$

for the third one we have

$$e\ell fyb = bx\ell fyb \in bAb$$

Since $\tau(a) \in L$ for any $a \in A$, $\tau(e\ell fyb) \in K$, then $b\tau(x\ell fy)b \in bKb \subseteq B$.
By Lemma 3.5 (3),

$$\tau(e\ell fyb) = \tau(bx\ell fyb) = b\tau(x\ell fy)b = \beta b \quad (3.8)$$

for some $\beta \in \mathbb{F}$
for the four one we have

$$e\ell fye\ell f = e\ell fybx\ell f = (e\ell f)yb(x\ell f) = 0$$

Since $f^*Lf = 0$, so $byf^*\ell e^*x\ell f = 0$. Then

$$(e\ell f)y(e\ell f) = e\ell fybx\ell f - byf^*\ell e^*x\ell f$$

$$= (e\ell fyb - (e\ell fyb)^*)x\ell f$$

$$= \tau(e\ell fyb)x\ell f$$

$$\beta b(x\ell f) = \beta e\ell f$$

$$\tau(e\ell fye\ell f) = \beta\tau(e\ell f) \quad (3.9)$$

Substituting equation 3.6 , 3.7, 3.8 and 3.9 in 3.5, we get that

$$cyc = (\lambda^2\alpha)b + (\lambda\alpha)\tau(e\ell f) + (\lambda\beta)b + \beta\tau(e\ell f)$$

$$= (\lambda^2\alpha + \lambda\beta)b + (\lambda\alpha + \beta)\tau(e\ell f)$$

$$= (\lambda\alpha + \beta)c$$

Therefore, $cKc = \mathbb{F}c$, B' is a point space
since B is a maximal point space, so $B = B'$
Therefore, B is a type one point space. ■

3.9 Theorem Suppose that $\boldsymbol{A}$ is simple with the orthogonal involution $*$ defined on it. If $\boldsymbol{p \neq 2, 3}$ and $\boldsymbol{A}$ is of dimensional greater than 16, Then every $\boldsymbol{J}$ −Lie $\boldsymbol{B}$ of $[\boldsymbol{K}, \boldsymbol{K}]$ is of the form $\boldsymbol{eKe^*}$ or $\boldsymbol{B}$ is a type one point space. where $\boldsymbol{e}$ is an idempotent in $\boldsymbol{A}$ such that $\boldsymbol{e^*e = 0}$.

Proof. Let $b \in B$, Then by Theorem 3.1, $\exists x \in K$ such that $b = bxb$.

Let $e = bx$. Then $e^* = (bx)^* = x^*b^* = xb$, since B is $J-$Lie, $b^2 = 0$, so $e^*e = xbbx = 0$. By Lemma 3.2, $bKb \subseteq B$

Suppose that $bKb \subseteq B$ is maximal with the property. Since

$$bKb = bxbKbxb \subseteq bxKxb = eKe^*;$$

$$eKe^* = bxKxb \subseteq bKb,$$

We have

$$eKe^* = bKb \subseteq B \qquad (3.10)$$

Next, we need to show that $B \subseteq eKe^*$

Let $e' = 1 - e$ and $e'^* = (1 - e)^* = 1 - e^*$, we have

$$b = 1b1 = (e + e')b(e^* + e'^*) = ebe^* + ebe'^* + e'be^* + e'be'^* \qquad (3.11)$$

First, we need to show that $e'Ke'^* = 0$

It remains to show that $e'Ke'^* = 0$. Assume to the contrary that $e'Ke'^* \neq 0$. Then $\exists c' \in K$ such that $z = e'c'e'^* \neq 0$. By Lemma 3.2 , $e'Ke'^* \subseteq B$, so $z \in B$. Let $c = b + z \in B$. In the view of Lemma 3.3(1), we have $c \neq 0$

First, we claim that $bKb \subseteq cKc$. Since $c \in B$, by Lemma 2.7, $cKc \subseteq B$. Take any $y \in K$. Then

$$ce^*yec = (b + z)e^*ye(b + z)$$

$$= be^*yeb + be^*yez + ze^*yeb + ze^*yez$$

Since $ez = e(e'c'e'^*) = 0$ *and* $ze^* = (e'c'e'^*)e^* = 0$,

$$ce^*yec = be^*yeb = bxbybxb = byb$$

so $ce^*Kec = bKb$. *As* $ce^*Kec \subseteq cKc$, *we get that*

$$bKb = ce^*Kec \subseteq cKc \qquad (3.12)$$

Next, we need to show that $zKz \subseteq cKc$. Take any $\ell \in K$, we have

$$ce'^*\ell e'c = (b + z)e'^*\ell e'(b + z)$$

$$= be'^*\ell e'b + be'^*\ell e'z + ze'^*\ell e'b + ze'^*\ell e'z \qquad (3.13)$$

By computing mutually each term, we get that

$be'^{*}\ell e'b = b(1-e^{*})\ell(1-e)b = b\ell b - b\ell eb - be^{*}\ell b + be^{*}\ell eb.$

$= b\ell b - b\ell bxb - bxb\ell b + bxb\ell bxb = b\ell b - b\ell b - b\ell b + b\ell b = 0$ (3.14)

$be'^{*}\ell e'z = b(1-e^{*})\ell(1-e)z = b\ell z - b\ell ez - be^{*}\ell z + be^{*}\ell ez$

$= b\ell z - bxb\ell z = b\ell z - b\ell z = 0$ (3.15)

$ze'^{*}\ell e'b = z(1-e^{*})\ell(1-e)b = z\ell b - z\ell eb - ze^{*}\ell b + ze^{*}\ell eb = z\ell b - z\ell b = 0$ (3.16)

$ze'^{*}\ell e'z = z(1-e^{*})\ell(1-e)z = z\ell z - z\ell ez - ze^{*}\ell z + ze^{*}\ell ez = z\ell z$ (3.17)

By substituting equation 3.14, 3.15, 3.16 and 3.17 in 3.13, we get that $ce'^{*}\ell e'c = z\ell z$. Since $\ell \in K$, by Lemma 3.2, $e'^{*}\ell e \in K$, so

$zKz = ce'^{*}Ke'c \subseteq cKc$ (3.18)

Recall that $z = e'c'e'^{*} \in K$. By Theorem 3.1, $\exists k \in K$ such that $z = zkz \in zKz$. By equation 3.18, we get that $z \in zKz \subseteq cKc$. But $z \notin bKb \subseteq cKc$, a contradiction. Therefore,

$e'Ke'^{*} = 0$ (3.19)

Therefore, $e'be'^{*} = 0$. Now we have to consider to two cases depending on eKe'^{*} whether it is zero or not

If $eKe'^{*} = 0$, then $(e'be^{*})^{*} = eb^{*}e'^{*} = -ebe'^{*} \in eKe'^{*}$

substituting in equation (3.11), we get that

$$b = ebe^{*} + ebe'^{*} - ebe'^{*} + e'be'^{*}$$

$$= ebe^{*} \in eKe^{*}$$

Therefore, $B = eKe^{*}$.

Suppose now that $eKe'^{*} \neq 0$. Then $\exists k \in K$ such that $w = eke'^{*} \neq 0$. Since

$$w^{*}Kw = (eke'^{*})^{*}K(eke'^{*})$$

$$= e'k^{*}e^{*}Keke'^{*} \subseteq e'Ke'^{*} = 0,$$

By Lemma 3.5 (3), rank $w \leq 1$, so $rank(w) = 0$ or $rank(w) = 1$. Thus, $rank(w) = 1$ (because $w \neq 0$).

Hence, $dim w(V)$ must be one, fix any $v_0 \in V$ such that $w(V) = \mathbb{F}v_0$.

Let $v \in V$ such that

$w(v) = v_0$. *(3.20)*

$$V = Im(w) + Ker(w)$$

$$= \mathbb{F}v + Ker(w)$$

Let $w' = e\ell e'^* \in eBe'^*$ be a non-zero transformation. Then

$$0 = e'(\ell(e^*Ke)k + k(e^*Ke)\ell)e'^*$$
$$= e'\ell(e^*Ke)ke'^* + e'k(e^*Ke)\ell e'^*$$
$$= (e'\ell e^*)K(eke'^*) + (e'ke^*)K(e\ell e'^*)$$
$$= w'^*Kw + w^*Kw'$$

If $u \in Ker(w)$, then

$$0 = \psi(0(v), u) = \psi((w'^*Kw + w^*Kw')(v), u)$$
$$= \psi(w'^*Kw(v), u) + \psi(w^*Kw'(v), u)$$
$$= \psi(Kw(v), w'(u)) + \psi(Kw'(v), w(u))$$

Since $u \in Ker(w)$, so $w(u) = 0$.

$$= \psi(Kw(v), w'(u))$$

By Lemma 3.5 (1), $w'(u) \in \mathbb{F}v_0$. Now either $w'(u) = 0$ or $w'(u) \neq 0$ for all $u \in Ker(w)$

If $w'(u) = 0$ for all $u \in Ker(w)$, then $Ker(w) \subseteq Ker(w')$

But $\mathrm{d}im(w(v)) = \mathrm{d}im(w'(v)) = 1$, so $Ker(w) = Ker(w')$

Suppose now that $w'(u) \neq 0$ for some $u \in Ker(w)$, then $\mathrm{I}m(w') = \mathbb{F}v_0 \subseteq \mathrm{I}m(w)$.

Since both have dimension 1, so $\mathrm{I}m(w') = \mathrm{I}m(w) = \mathbb{F}v_0$.

Then by Theorem 3.6, B' is a J −Lie.

Now, we need to show that $B = B'$, by Theorem 3.7,

$$e(V) = eKe^*K(v_0^{\perp})$$

and

$$eBe'^*(V) = e(V) \qquad (3.21)$$

we claim that $eB'e'^*(V) \subseteq eBe'^*$

we have $B' = eKe^* + \tau(eKf)$

$$eB'e'^* = e(eKe^* + \tau(eKf))e'^*$$

$$= eKe^*e'^* + eKfe'^* - (eKf)^*e'^*$$

$$= eKfe'^* - f^*Ke^*e'^* = eKfe'^*$$

Since $e^*e'^* = 0$. Recall that $fe'^* = f$,

$$eB'e^* = eBf$$

Let $e\ell f \in eKf$

$$e\ell f(v) = e\ell zw(v) = e\ell z(v_0) \in eKK(v_0) = eK(v_0^{\perp}) = e(v_0) \in e(V)$$

because $(w(v) = v_0)$. By equation (3.21), $e(V) = eBe'^*$

$e\ell f(v) \in eBe'^*(V)$. Therefore $\exists w' \in eBe'^*$ such that $e\ell f(v) = w'(v)$.

Since $V = \mathbb{F}v_0 + Ker(w)$ and $Ker(f) = Ker(w) = Ker(w')$ fore equation (3.1), and $w' \in eBe'^*$,

for any $e\ell f \in eKf = eB'e'^*$, there exist $w' \in eBe'^*$ such that $e\ell f = w' \in eBe'^*$

Then $eB'e'^* \subseteq eBe'^*$ and $B' \subseteq B$. Therefore $B' = B$.

There exist idempotent $(e + f^*)$ such that $B = (e + f^*)K(e + f^*)^*$.

Now, when $bKb = \mathbb{F}b$, by Theorem 3.8, $B' = B$ is a type one point space.

Suppose that $Im(w') = Im(w) = \mathbb{F}v_0$ for any $w' \in eBe'^*$, we need to show that B is a type one point space

Recall that $wzw = w$, $z = e'^*z'e$

Let

$$f = wz = eb'e'^*z'e$$

Then

$$fe' = eb'e'^*z'e(1 - e) = 0$$

and

$$e'f = (1 - e)eb'e'^*z'e = 0$$

$$f^2 = (eb'e'^*z'e)(eb'e'^*z'e) = eb'e'^*z'e = f$$

$ef = eeb'e'^*z'e = eb'e'^*z'e = f$ and $fe = eb'e'^*z'ee = f$

Since $rank(f) = 1$, so $rank(f^*) = 1$

Recall that $f^*Kf = fKf^* = 0$. we have $Im(w) = Im(f)$

for any $w' \in eBe'^*$, we have $Im(w) = Im(f) = Im(w')$

we have going to prove that there exist point space $B' = eKe^* + \tau(fKe'^*)$ such that $B = B'$

First, we claim that $fw' = w'$ for any $w' \in eBe'^*$

Let $u \in Ker(w')$, then $fw'(u) = 0$

so $Ker(w') \subseteq Ker(f(w'))$, therefore $Ker(w') = Ker(f(w'))$ (co-dimension 1)
Since $\mathrm{I}m(w') = \mathrm{I}m(fw')$, so $fw' = w'$ for any $w' \in eBe'^{*}$.
Second, we claim that

$$B \subseteq B' = eKe^{*} + \tau(fKe'^{*})$$

take $K \in B$, then

$$\mathrm{K} = \mathrm{eKe^{*}} + \mathrm{eKe'^{*}} + \mathrm{e'Ke^{*}} + \mathrm{e'Ke'^{*}}$$

$$\mathrm{K} = \mathrm{eKe^{*}} + \tau(\mathrm{eKe'^{*}})$$

$$\mathrm{K} = \mathrm{eKe^{*}} + \tau(\mathrm{w'})$$

Since $fw' = w'$, so

$$K = eKe^{*} + \tau(feKe'^{*})$$

because $fe = f$. For all $K \in B$, we have

$$K = eKe^{*} + \tau(fKe'^{*})$$

$$B \subseteq B' = eKe^{*} + \tau(fKe'^{*})$$

Now, we claim that B is a point space, that is $bKb \neq \mathbb{F}b$. Then

$$\mathrm{eKe^{*}K(v_0)} \subseteq \mathrm{eKe'^{*}(v_0)} \subseteq \mathrm{fKe'^{*}(v)} = \mathbb{F}v_0$$

but

$$eKe'^{*}K(v_0) = e(V) \neq \mathbb{F}v_0$$

because $\mathrm{r}anke(V) > 1$.
Finally, we claim that $B' = eKe^{*} + \tau(fKe'^{*})$ is point space
By using equation (3.4), and our assume that $bKb = \mathbb{F}b$, we have

$$\mathrm{B'} = \mathrm{eKe^{*}} + \tau(\mathrm{fKe'^{*}})$$

$$= \mathrm{bxKxb} + \tau(\mathrm{fKe'^{*}})$$

$$= \mathrm{bKb} + \tau(\mathrm{fKe'^{*}}) = \mathbb{F}\mathrm{b} + \tau(\mathrm{fKe'^{*}})$$

for any $c' \in B', \exists \ell \in K, \lambda \in \mathbb{F}$ such that

$$\mathrm{c'} = \lambda b + \tau(f\ell e'^{*})$$

for all $y \in K$, we have

$$c'yc' = (\lambda b + \tau(f\ell e'^{*}))y(\lambda b + \tau(f\ell e'^{*}))$$

$$= \lambda^2 byb + \lambda by\tau(f\ell e'^{*}) + \lambda\tau(f\ell e'^{*})yb + \tau(f\ell e'^{*})y(f\ell e'^{*})$$

$= \lambda^2 byb + \lambda(by)\tau(f\ell e'^*) + \lambda\tau(f\ell e'^*)(by)^* + \tau(f\ell e'^*)y\tau(f\ell e'^*)$

By Lemma 3.4 (3),

$= \lambda^2 byb + \lambda\tau(byf\ell e'^*) + \lambda\tau(f\ell e'^*(by)^*) + \tau(f\ell e'^* yf\ell e'^*)$

$-f\ell e'^* y(f\ell e'^*)^* - (f\ell e'^*)^* yf\ell e'^*$

$= \lambda^2 byb + \lambda\tau(byf\ell e'^*) + \lambda\tau(f\ell e'^* yb) + \tau(f\ell e'^* yf\ell e'^*)$

$-f\ell e'^* ye'\ell^* f^* - e'\ell^* f^* yf\ell e'^*$

Since $fKf^* = f^*Kf = 0$

$c'yc' = \lambda^2 byb + \lambda\tau(byf\ell e'^*) + \lambda\tau(f\ell e'^* yb) + \tau(f\ell e'^* yf\ell e'^*)$ (3.22)

we need to calculate each term
Since $bKb = \mathbb{F}b$, so

$byb = \alpha b$ *(3.23)*

$\tau(byf\ell e'^*) = \tau(byef\ell e'^*) = \tau(bybxf\ell e'^*) = \tau(\alpha bxf\ell e'^*)$

$= \tau(\alpha(ef\ell e'^*) = \alpha\tau(ef\ell e'^*)$ *(3.24)*

For the third one we have

$f\ell e'^* yb = ef\ell e'^* yb = bxf\ell e'^* yb \in bAb$

Since $\tau(a) \in L$ for any $a \in A$, so $\tau(xf\ell e'^* y) \in K$,then $b\tau(xf\ell e'^* y)b \in bKb \subseteq B$
By Lemma 3.5 (3),

$\tau(f\ell e'^* yb) = \tau(bxf\ell e'^* yb) = b\tau(xf\ell e'^* y)b = \beta b$ (3.25)

Lastly, we have

$f\ell e'^* yf\ell e'^* = f\ell e'^* \text{yef}\ell e'^* = f\ell e'^* ybxf\ell e'^*$

$= (f\ell e'^*)yb(xf\ell e'^*)$

Since $f^*Lf = 0$, so $bye'\ell f^* xf\ell e'^* = 0$. Then

$(f\ell e'^*)y(ef\ell e'^*) = f\ell e'^* ybxf\ell e'^* - bye'\ell f^* xf\ell e'^*$

$$= (f\ell e'^{*}yb - (f\ell e'^{*}yb)^{*})xf\ell e'^{*}$$

$$= \tau(f\ell e'^{*}yb)xf\ell e'^{*} = \beta b(xf\ell e'^{*})$$

$$\Longrightarrow \tau(f\ell e'^{*}yf\ell e'^{*}) = \beta\tau(f\ell e'^{*}) \qquad (3.26)$$

Substituting equation 3.23, 3.24, 3.25 and 3.26 in equation 3.22. We get that

$$c'yc' = (\lambda^2\alpha)b + (\alpha\lambda)\tau(ef\ell e'^{*}) + (\lambda\beta)b + \beta\tau(f\ell e'^{*})$$

$$= (\lambda^2\alpha + \lambda\beta)b + (\alpha\lambda + \beta)\tau(f\ell e'^{*})$$

$$= (\lambda\alpha + \beta)(\lambda b + \tau(f\ell e'^{*}))$$

$$c'yc' = (\lambda\alpha + \beta)c'$$

Therefore, $c'Kc' = \mathbb{F}c'$
B' is point space and $B \subseteq B'$ but B is maximal. Therefore

$$B = B'$$

B is a type one point space.■

4 Conclusion

Every Jordan-Lie inner ideals of the orthogonal Lie algebras is either $B = eKe^{*}$ or B is a type one point space. one can find an idempotent $e \in A$ such that this inner ideal can be written in the form eKe^{*}. We study the relationship between these algebras and their corresponding Lie ones. Also study Jordan-Lie inner ideals of these Lie algebras. proved that every Jordan-Lie inner ideal of the orthogonal Lie algebra of an associative algebra (finite dimensional) is generated by an idempotent $e \in A$ with the property $e^{*}e = 0$.

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Article submitted 4 July 2022. Published as resubmitted by the authors 1 August 2022.

New Types of Continuous Function and Open Function

https://doi.org/10.31185/wjcm.Vol1.Iss2.35

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Abstract—In this paper, we continue to study the properties of the relation with some type of open sets, and we introduce α-continuous function, semi-continuous function, α^*-continuous function, and α^{**}-continuous function are studied and some of their characteristics are discussed. In this work, we need to introduce the concepts of function, especially the inverse function to find all continuous function, so we want to prove some examples, theorems, and observations of our subject with the help of new concepts for the alpha-open sets of sums to make it easier for us to find a relationship between these formulas as well as the converse relationship has been studied and explained with illustration many examples. Hence, reaching to get a relationship (continuous, α-continuous, semi α-continuous) function at new condition.

Keywords— (semi-continuous, α-continuous, semi α-continuous, α^*-continuous, α^{**}-continuous) function

1 Introduction

Open sets and closed sets play a key role in constructing topological space, which is why scientists and researchers in the field of mathematics paid great attention to them, and used new patterns as synonyms for open and closed sets. Our studies focus on continuous functions of the alpha type.it is known that the continuity of functions is evident from the concept of open and closed function s according to the following criterion. Let the function be defined from the topological space to another one, $f:(X,\tau_x) \to (Y,\tau_y)$. Then the function is continuous if and only if the invers image of each set is open or closed in the second space is also open or closed in the first space.

2 Preliminaries

2.1 Definition [1]

If f:(X, T_X)→ (Y, T_Y) be two topological space. Then f is named $\boldsymbol{\alpha}$**-continuous** function if and only if, for each $\boldsymbol{A}$ ***is open*** set *in* $\boldsymbol{Y}$.

Thus $\boldsymbol{f^{-1}}$ **(A)** is **α-open** Set in (X, T_X).

2.2 Definition [2]

If $f : (X, T_X) \rightarrow (Y, T_Y)$ be two topological space**,** thus f is termed ***semi*-continuous** function. When A *is open set in* Y, $f^{-1}(A)$ in(Y,τ_x) thus $f^{-1}(A)$ *is* ***semi*-open** set in (X, τ_x). Such that $(f^{-1}(A) \subseteq Cl\ Int\ f^{-1}(A))$

2.3 Theorem [2]

Each *contiuous function* is *semi-contiuous* function.
Proof :
Let $f$be *contiuous*, there exists $f^{-1}(A)$ open in (Y,τ_Y)
Therefore $f^{-1}(A)$is open in(X, τ_x). since (**every open set is *semi*-open set**)
Then f is *semi*-continuous.

The opposite of the previous theorem does not have to be true and the following Example illustrates this.

2.4 Example

If $X = \{0,2,4,6\}$, $T_x = \{\emptyset, \{4\}, \{0,2\}, \{0,2,4\}$. Define in$(X, T_x$)space,
And let $Y = \{1,3,5\}$, $T_y = \{\emptyset, \{5\}, \{3\}, \{1,3\}, \{5,3\}, Y\}$. Define in Y space,
Then $f: X \rightarrow Y$; $f(x_1) = f(x_2) = 3$, $f(x_3) = 5$, $f(x_4) = 1$,,
Clearly ; open sets of space $Y : \{\emptyset, \{5\}, \{3\}, \{1,3\}, \{5,3\}, Y\}$.
And also *semi*-open sets of space X :$SO(X, T_x)$
$\{\emptyset, \{4\}, \{0,2\}, \{0,2,4\}, \{0,2,6\}, \{4,6\}, X\}$.
That is perfect f is *semi*-continuous,
But $\boldsymbol{f}$ is not continuous because ;
$\boldsymbol{f^{-1}(\{1,5\}) = \{4,6\} \notin T_x}$**.**

2.5 Remark

Let $f : (X, T_x) \rightarrow (Y, T_y)$;
Continuous → $\boldsymbol{\alpha}$**-continuous** → ***semi*** $\boldsymbol{\alpha}$**-continues**. But the convers is not Right in general.

2.6 Theorem [3]

If (X, T_X) too (Y, T_Y) are a $topological\ spaces$ and if $f: X \rightarrow Y$ be α-$continuous$ function, then f is $semi$-$continuous$.
Proof :
Since f is α-continuous function. Thus $f^{-1}(A) \subseteq Int\ Cl\ Int f^{-1}(A)$.
By (proposition **let $f: X \rightarrow Y$ is α -continuous if and only if each open Set A of Y $f^{-1}(A) \subseteq Int\ Cl\ Int\ f^{-1}(A)$.)**
Obviously $Int\ Cl\ Int\ f^{-1}(A) \subseteq Cl\ Int\ f^{-1}(A)$,
So $f^{-1}(A) \subseteq Cl\ Int\ f^{-1}(A)$. hence f is $semi$-continuous.
(**By, let (X, T_X) and (Y, T_Y) be two $topological\ Space$,**
Thus $f: X \rightarrow Y$ is semi-$continuous$ if every $open\ set\ A in\ Y$,
$f^{-1}(A) \subseteq cl\ Int\ f^{-1}(A)$).

The convers of theorem **(2.1.6.)** is not certainly true in general. To get this, We offer the previous counter example is given.

2.7 Example

If $X = \{7,8,9\}$, $T_x = \{\emptyset, \{7\}, \{8\}, \{7,8\}, X\}$,
$T_y = \{\emptyset, \{7\}, \{8,9\}, X\}$,Therefore the self-function, since $X = Y$.
$f: (X, T_x) \rightarrow (X, T_y)$ be $semi$-continuous,
However no α-continuous.
Every continuous $function\ is\ \alpha - continuou$s function, so $it\ is$ semi α-$continuous$, On the other hand the convers is false in universal.

2.8 Example

If $X = \{0,1,3,5\}$, and $T_x = \{\emptyset, \{0\}, X\}$, let $y = \{2,4,6\}$, $T_y = \{\emptyset, \{2\}, Y\}$.
The α-open sets define on space X are ;
$T^{\alpha}_{(x)} = T_x \cup \{\{0,1\}, \{0,3\}, \{0,5\}, \{0,1,3\}, \{0,1,5\}, \{0,3,5\}\}$,
The α-open sets define on space Y are ; $T^{\alpha}_{y} = \{\emptyset, \{2\}, \{2,4\}, \{2,6\}, Y\}$,
If $f: X \rightarrow Y$, define are $f(x_1) = f(x_2) = 2$, $f(x_3) = 4, f(x_4) = 6$.
Since f is α-$continuous$ function, however it is not $continuous$ function.
Because $\{2\}$ is open in space Y, as $f^{-1}(\{2\}) = \{0,1\}$,
But $\{0,1\}$is not open in Space X.
To fined semi α-continuous ; $S\alpha O(X) = T^{\alpha}_{x}$, and $S\alpha O(Y) = T^{\alpha}_{y}$
Since **f is $semi\ \alpha$-continuous** function, but it is **not continuous**, $\{2\}$is open,
However $f^{-1}(\{2\}) = \{0,1\}$, is not open in T_x.
Each α-continuous function is $semi\ \alpha$-continuous, however convers is not True In General.

2.9 Example

Give $X = \{4,6,8\}$, $T_x = \{\emptyset, \{4\}, \{6\}, \{4,6\}, X\}$,
Then the $\boldsymbol{\alpha}$-***open*** *sets in space* X ; $T_x^{\alpha} = T_x$,
As well as the***semi*** $\boldsymbol{\alpha}$-***open*** *sets in space* $X, S\alpha O(X) = T_{(x)}^{\alpha} \cup \{\{6,8\},\{4,8\}\}$.
Thus function define by *identity* ; $f(x_1) = 4$, $f(x_2) = f(x_3) = 6$,
Therefore function is *semi* α-continuous , but it is not α-continuous.
because, $\{6\}$is **open** set, then $f^{-1}(\{w\}) = \{w, e\} \notin T_x^{\alpha}$.
To find a *semi* α-continuous ; $S\alpha O(X) = T_x^{\alpha} \cup \{\{w,e\},\{q,e\}\}$,
Then $\boldsymbol{f}$ **is** ***semi*** $\boldsymbol{\alpha}$**- continuous**, however it is **not** $\boldsymbol{\alpha}$**-continuous**.

3 Continuity and Function relationship

3.1 Theorem [4]

let $f : (X, \tau_x) \to (Y, \tau_y)$ and $g : (Y, \tau_y) \to (Z, \tau_z)$, are equally continuous function then the composition $gof : (X, \tau_x) \to (Z, \tau_z)$ is continuous.
Proof :
If $M \in \tau_Z$, then $g^{-1}(M) \in \tau_y$, (by g is continuous). Such that $g^{-1}(M) \subseteq Y$
Therefore f^{-1} ($g^{-1}(M)) \in \tau_x$. (by f be continuous)
And $(f^{-1}og^{-1})(M) \in \tau_X$,
Thus $(gof)^{-1}(M) \in \tau_x$, (by $(gof)^{-1} = f^{-1}og^{-1}$)
Then gof is composition.(continuous)

3.2 Remark [4]

The composition of finite number of continuous function is continuous.
Explain : the composition of four or seven or fifty continuous function is continuous (if f, g, h, k are continuous, so $kohogof$ is continuous…).
If f:X→Y, g:Y→Z, *are* α-*continuous*, and the arrangement function,
$\boldsymbol{gof}$ **is not necessary α-continuous.**

3.3 Example

Let $X = \{0,3,5,7\}$, $T_x = \{\emptyset, \{5\}, \{0,5\}, \{0,3,5\}, X\}$.
And let $Y = \{2,4,6\}$, $T_y = \{\emptyset, \{6\}, Y\}$. If α-open sets in space X ;
$\boldsymbol{T_{(x)}^{\alpha}} = T_x \cup \{\{3,5\},\{5,7\},\{5,3,7\},\{0,5,7\}\}$. .And the α-open sets in space Y ;
$\boldsymbol{T_{(y)}^{\alpha}} = T_y \cup \{\{2,6\},\{4,6\}\}$.So f:X→Y ; $(x_1) = f(x_2) = 2, f(x_3) = f(x_4) = 4$.
And g:Y→X ;g (y_1)=g(y_2)= 5, g (y_3)= 0. Therefore f, g are α-continuous.
$gof : X \to X$, $gof(0) = gof(3) = 5$, $gof(5) = gof(7) = 0$.The gof,
is not α- continuous because; $\{5\}$is *open set* of *space* X ,
$(gof)^{-1}(5) = \{0,3\}$, but $\{0,3\}$be not α-*open* of *space* X.

3.4 Definition [5]

If f: X→ Y. Then f is named $\boldsymbol{\alpha^*}$**-continuous**,
for Each N is $\boldsymbol{\alpha}$-$\boldsymbol{open}$ set of $\boldsymbol{Y},$ thus $\boldsymbol{f^{-1}}$ $(\boldsymbol{N})$be $\boldsymbol{\alpha}$-$\boldsymbol{open}$ set of X.

3.5 Theorem [1]

A function $f: X \to Y$. Therefore the following statement are equivalent .
a) f is $semi$ α-$continuous$.
b) f is $semi$ α-$continuous$ at each point $x \in X$.
$\boldsymbol{Proof}$:
a) ⟹ b)
let $f : X \longrightarrow Y$ is a $semi$ α-continuous.
And $x \in X$, N is an $open$ set of Y having $f(x)$.
Then $x \in f^{-1}(N)$. also f is $semi$ α-continuous.
So $M = f^{-1}(N)$ is $semi$ α-$open$ set in X holding (x).
therefore $f(M) \subset N$.
b)⟹ a)
if f : X⟶ Y is a $semi$ α-continous for all point in X.
And N $open$ set in Y. Let $x \in f^{-1}(N)$.
Then N is $open$ set in Y containing $f(x)$.
By (b), at hand is $semi$ α-$open$ set M of X containing x.
Since $f(x) \in f(M) \subseteq N$. Therefor M ⊆ f^{-1}(N).
Hence f^{-1} (N)= ∪{M :x∈ f^(-1) (N) }.
Then f^{-1} (N) is $semi$ α-$open$ in X.

3.6 Remark

The notions of continuity and α^*-continuity are independent,.

3.7 Example

If $X = \{1,2,3,4\}$, $T_X = \{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}, X\}$, $T^{\alpha}_{(X)} = T_X$.
And $Y = \{5, 6\ 7\}, T_Y = \{\emptyset, \{5\}, Y\}$,
$T^{\alpha}_{(Y)} = T_Y \cup \{\{5,6\}, \{5,7\}\}$.
If f :X→Y by f (x_1)= 5 , f (x_2)= 6 , f (x_3)=f $(x_4$)= 7.
Then f is **continuous,** However it is not $\boldsymbol{\alpha^*}$**-continuous**.
Since $\{5,6\} \in \tau^{\alpha}_{(Y)}$, but $f^{-1}\{5,6\} = \{1,2\} \notin T^{\alpha}_{(X)}$.
Hence f is **continuous**. And f is **not** $\boldsymbol{\alpha^*}$**-continuous** function.

3.8 Example

If $X = \{1,2,3,4\}$, $T_X = \{\emptyset, \{1\}, X\}$,
$T^{\alpha}_{x} = T_x \cup \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$.

$Y = \{5,6,7\}$, $T_y = \{\emptyset, \{5\}, Y\}$,
$T_y^\alpha = T_Y \cup \{\{5,6\},\{5,7\}\}$,
If $f: X \to Y$, with f (x_1)= f (x_2)= 5 , f (x_3)= 6 , f (x_4)= 7
Hence f is α^*-continuous, but it's not continuous, Because{5}is open set in Y,
However $f^{-1}(\{5\}) = \{1,2\}$ is not open in X.
As a result $\boldsymbol{f}$ is $\boldsymbol{\alpha^*}$**-continuous**, however $\boldsymbol{f}$ is **not continuous** .

3.9 Proposition [3],[1]

1. A function $f : (X, T_x) \to (Y, T_y)$ is an open, continuous and bijective, then f is α^*-continuous.
2. A meaning $f : (X, T_x) \to (Y, T_y)$ *are* α^**-continuous* iff, $f : (X, T_x^\alpha) \to (Y, T_y^\alpha)$ *are continuous*.

Proof :

Let $E \in T_x^\alpha$, to prove f^{-1}(E)∈ T_x^α, Then $f^{-1}(E) \subseteq Int\ Cl\ Int\ f^{-1}$(E)

If $x \in f^{-1}$(E)⟹ f (x) ∈(E). and f(x)∈ $Int\ Cl\ Int\ E$ (since $E \in T_y^\alpha$) .
And so, there occurs N *open set* of Y. Since $f(x) \in N \subseteq Cl\ Int\ E$.
And $x \in f^{-1}(N) \subseteq f^{-1}(cl\ Int\ E)$, then $f^{-1}(cl\ Int\ E) \subseteq cl(f^{-1}(IntE))$.
(then $\boldsymbol{f^{-1}}$is continuous, which is same to $\boldsymbol{f}$ is open and bijective)
Thus $x \in f^{-1}(N) \subseteq Cl(f^{-1}(Int\ E))$.
Since $x \in f^{-1}(N) \subseteq Cl(f^{-1}IntE)) \subseteq Cl(Int(f^{-1}(E)))$, (**$\boldsymbol{f}$ is continuous**)
Therefore $x \in f^{-1}$(N)⊆ $Cl(Intf^{-1}$(N),
But $f^{-1}(N)$ is open set in X, (**$\boldsymbol{f}$ is continuous**)
Thus $x \in Int\ Cl$ (Int$(f^{-1}(N))$, ,As a result $f^{-1}(N) \subseteq Int\ Cl$ Int$(f^{-1}(N))$,
Then f^{-1}(N)∈ T_x^α. therefore f is α^*-continuous function.
To prove (2) is obviously.

3.10 Remark [1]

The concepts of continuity and *semi* α -continuity are independent,
Example.

3.11 Example

If $X = \{0,2,4,6\}, T_x = \{\emptyset, \{0\}, \{0,4\}, \{2,4,6\}, X\}$. Thus $T_x^\alpha = T_X$,
Let $y = \{7,8,9\}$, $T_y = \{\emptyset, \{7\}, Y\}, T_y^\alpha = T_y \cup \{\{7,8\},\{7,9\}\}$.
Define f:X→Y, by $f(x_1) = 7$, $f(x_2) = 8$, $f(x_3) = f(x_4) = 9$.
It is simply seen, f be continuous, then be no *semi* α^*-continous, then
$\{7,8\} \in S\alpha O(Y)$, but $f^{-1}(\{7,8\}) = \{0,2\} \notin S\alpha O(X)$.
Therefore f is continuous however it is not *semi* α^*-continuous.

3.12 Example

Let us equip that the sets X and Y of the above example with topologies,
$T_x = \{\emptyset, \{0\}, X\}, T_x^\alpha = T_x \cup \{\{0,2\}, \{0,4\}, \{0,6\}, \{0,2,4\}, \{0,2,6\}, \{0,4,6\}\}$
$S\alpha O$(X)= T_x^α, T_y ={$\emptyset$,{7},Y}, $T_y^\alpha = T_y \cup \{\{7,8\}, \{7,9\}\}, S\alpha O(Y) = T_y^\alpha$,
Then describe f:X→Y, by $f(x_1) = 7,\ f(x_2) = 8,\ f(x_3) = f(x_4) = 9$.
It is simply told that f is $semi\ \alpha^*$-continuous, but it is not continuous,
Because {7}is $open$ of Y. then $f^{-1}(\{7\}) = \{0,2\}$be $open$ of X.
Therefore f is $semi\ \alpha^*$-continuous, however it is not continuous.

3.13 Definition [17]

If $f\ :\ X\ \rightarrow\ Y$ is a $function$, thus $f\ is$ termed $\boldsymbol{\alpha^{**}}$***-continuous*** if and only if, For each $\boldsymbol{N}$ $\boldsymbol{\alpha}$***-open*** set of $\boldsymbol{Y}$, thus $\boldsymbol{f^{-1}}(\boldsymbol{N})$ be ***open*** set of $\boldsymbol{X}$.

3.14 Example

If $X = \{5,3,1,0\}$, $T_X = \{\emptyset, \{5,1\}, \{5,3,1\}, X\}$,
$T_x^\alpha = T_X \cup \{5,3,0\}$. With $f\ is$ Identity function.
$f(x_1)\ =\ f(x_2)\ =\ 3, f(x_3) = f(x_4) = 1.1$. Thus $f$be α-$open$ set in Y,
Because {5,3,1} is $opf(x_1)\ =\ f(x_2)\ =\ 3, f(x_3) = f(x_4) = 1$then
of X, $f^{-1}(\{5,3,1\}) = X$ an open in X.
Hence f is α-open and open function. So f is α^{**}-continuous.

4 Conclusion

For topological space, through our study between the relations, continuous, alpha-continuous, and semi-alpha-continuous. we get a direct representation of their abbreviation ; the relationship continuous→alpha-continuous→semi-alpha continous. And prove ; $f : (X, T_x) \rightarrow (Y, T_y) are$ alpha star-continuous $\Leftrightarrow f : (X\ , T_x^\alpha) \rightarrow (\ Y\ , T_y^\alpha)$ are continuous.

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Article submitted 21 June 2022. Published as resubmitted by the authors 1 August 2022.

Topological Spaces F_1 And F_2

https://doi.org/10.31185/wjcm.Vol1.Iss2.36

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Abstract—The aim of This work is to present new types of spaces, which are F_1 space and F_2 space, based on definition of new types of open sets, which is the feebly open set. we study there the basic properties and also obtained the effects between F_1 space and F_2 space and with the know space $T_{1/2}$ space and with closed and open sets.

Keywords— F_1 space and F_2 space

1 Introduction

One of a topology's most important and fascinating concepts is the idea of feebly separation characteristics. In 1963, N.levin[1] proposed concept of a semi-open set. S.N Maheshwari and R. Prasad[2], used a semi-open set to characterize and investigate novel division known as semi-detachment aphorisms. In 1975, N Levine characterized the idea of new type of topological space called $T_{1/2}$ in 1970 [3] (i.e. the space where the closed sets and summed up sets classes meet). Maheshwari S. N and Tapiu[4] initiated the study of feebly open in 1978. In 2019[5], Ail Khalaf Hussain Al-Hachami presented the idea of some feebly separation properties, it is demonstrated that every feebly-T_1 is semi-T_1 and every feebly-$T_{1/2}$ is feebly-T_0. Aaad Aziz Hussan Abdulla in [6] presented the idea of semi-feebly open (sf-open) set. In 2021, Ail .Al kazaragy, Faik. Mayah and Ail Khala Hussain Al-Hachami [7]. introduced defined semi-θ -axioms. Zainab Awad Kadhum and Ail Khala Hussain[8] defied ii δ_g –closed set in topological spaces, it is demonstrated that each ii-$T_{3/4}$ space is ii-$T_{1/2}$ space "the goal of this study is to provide some characterizations of F_1 space and F_2 space ".

2 Basic definition

2.1 Definition[8]

A subset A of a topological space (X, τ) is called feebly open (f-open) set if there exists an open set U such that $U \subseteq A \subseteq \overline{U}^{s}$.

2.2 Definition [9]

Let (X, τ) be a topological space. A subset A of X is said to be g-closed if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is open set.

2.3 Definition[3]

Let (X, τ) be a topological space. A subset A of X is said to $T_{1/2}$ space if each g-closed set is closed set.

3 Characterization of F_1 space and F_2 space

3.1 Definition

Let (X, τ) be a topological space. A subset A of X is said to be
(1) F^*-closed set if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is f-open set.
i.e. $\forall\, U$ is f-open in X ($A \subseteq U \longrightarrow \overline{A} \subseteq U$)
(2) F^*-open set if the complement of A in X is F^*-closed set.

3.2 Example

Let $X = N$, the set of all natural numbers, $\tau = \mathrm{P}(x)$ be a topology defined on X.
Let $A = \{x \in N : x$ add number$\}$, then $\overline{A} = \{x \in N : x$ add number$\}$, thus the f-open sets contain A is A.
Hence $\overline{A} \subseteq A$, then A is F^*-closed set.

3.3 Proposition

Every closed set is F^*-closed set.

3.4 Remark

The converse [**Proposition (3.3)**] is not necessarily true as shown by the following example.

3.5 Example

Let $X = \{1, 2, 3\}$, $\tau = \{X, \emptyset, \{1, 3\}\}$ be a topology defined on X.
Let $A = \{1, 2\}$, then $\overline{A} = X$, thus the f-open sets contain A is only X
It is clear A is F^*-closed set but not closed.

3.6 Proposition

Every F^*-closed set is g-closed set.

3.7 Remark

The converse [**Proposition (3.6)**] is not necessarily true as shown by the following example.

3.8 Example

Let $X = \{1, 2, 3\}$, $\tau = \{X, \emptyset, \{1\}\}$ be a topology defined on X.
Let $A = \{1, 3\}$, then $\overline{A} = X$, implies the f-open sets contain A is X and $\{1, 3\}$. It is clear $\overline{A} \nsubseteq \{1, 3\}$
Hence A is not F^*-closed set but A is g-closed set since the open sets contain A is only X, $\overline{A} \subseteq X$.

3.9 Remark

g-closed set is F^*-closed set if every f-open set is open set.

3.10 Lemma

Let (X, τ) is a topological space, if every closed set is open set then
(1) every f-open set is f-closed set.
(2) every f-open (f-closed) set is open set (closed) set.

3.11 Theorem

Let (X, τ) be a topological space , then $\tau = f$ if and only if every subset of X is F^*-closed set, where f is the family of closed sets in X.
Proof.
$\Longrightarrow$ Let $\tau = f$ and $A \subseteq X$ and $A \subseteq O$, where O is f-open set in X.

Since $A \subseteq O$, then $\overline{A} \subseteq \overline{O}$, but $\overline{O} = O$.

Then $\overline{A} \subseteq O$, implies A is F^*- closed set.

$\Leftarrow$ Let every subset of X is F^*- closed set

Assume that $O \in \tau$ then O is f-open set

Since $O \subseteq O$, O is F^*- closed set

Hence $\overline{O} \subseteq O$ implies $\overline{O} = O$

Therefore $O \in f$

Thus $\tau \subset f$ (1)

Assume that $F \in f$, then $F^c \in \tau$

Hence F^c is f-open set

Since $F^c \subset X$, implies F^c is F^*- closed set

But $F^c \subseteq F^c$ and F^c is f-open set

So, $\overline{F^c} \subseteq F^c$, consequently $F^c \in f$

Therefore $F \in \tau$

Thus $f \subset \tau$ (2)

Then by (1) and (2), we have $\tau = f$.

3.12 Remark

Let (X, τ) is a topological space. If $\tau = f$ then every g-closed set is F^*-closed set.

3.13 Theorem

Let (X, τ) be a topological space and $A \subseteq Y \subseteq X$ and Y open set in X. If A is F^*- closed set in X then A is F^*- closed set in Y.

Proof.

Let $A \subseteq O$, O is f-open in X

Then $A \cap Y \subseteq O \cap Y$

Since Y is open set in X.

Hence $O \cap Y$ is f-open in Y.

Since A is F^*- closed set in X.

So, $\overline{A} \subseteq O$.

Therefore $\overline{A} \cap Y \subseteq O \cap Y$.

But $\overline{A}_Y = \bar{A} \cap Y$, such that $\overline{A}_Y$ is closure of A in Y.

Thus A is F^*- closed set in Y

3.14 Theorem

If $A \subseteq Y \subseteq X$ such that Y is open and closed in X, A is F^*- closed set in Y then A is F^*- closed set in X.

Proof.

Let $A \subseteq O$ and O is f-open in X

Then $A \cap Y \subseteq O \cap Y$

Since $A \subseteq Y$ implies $A \subseteq O \cap Y$

But Y is open set in X

Hence $O \cap Y$ is f-open in Y

Since A is F^*- closed set in Y

Then $\overline{A}_Y \subseteq O \cap Y$

We know $\overline{A}_Y = \overline{A} \cap Y$

Since $A \subseteq Y$ implies $\overline{A} \subseteq \overline{Y}$

But Y is closed set in X

Therefore $\overline{A} \subseteq Y$

So, $\overline{A} \subset O \cap Y$

Then $\overline{A} \subseteq O$

Thus A is F^*- closed set in X.

3.15 Definition

A topological space X is called F_1 space if and only if every g-closed set in X is F^*- closed set.

3.16 Example

Let X = {1, 2, 3}, τ = {∅, X, {1}, {3}, {1, 3}} be a topology defined on X.
Then g-closed on X are {∅, X, {2}, {1, 2}, {2, 3}},
F^*- closed set on X are {∅, X, {2}, {1, 2}, {2, 3}}.
It is clear every g-closed set is F^*-closed set.

3.17 Proposition 3.17

Let X be a topological space, if X is $T_{\frac{1}{2}}$ space, then X is F_1 space.

Proof.
Assume that A is g-closed set in X.
Since X is $T_{\frac{1}{2}}$ space, implies A is closed set.
Therefore, A is F^*- closed set [**Proposition (3.3)**].
Then X is F_1 space.

3.18 Remark

The converse [**Proposition (3.17)**] is not necessarily true as shown in the Following example.

3.19 Example

Let X = {1, 2, 3, 4}, τ = {∅, X, {3}, {1, 2}, {1, 2, 3}} be a topology defined on X.
The closed sets are {∅, X, {1, 2, 3}, {3, 4}, {4}}
g-closed sets are {∅,X,{2,3,4}, {1, 3, 4}, {2, 4}, {1, 4}, {1, 2, 4}, {3, 4},{4}}.
F^*- closed sets are {∅,X,{2,3,4},{1, 3, 4}, {2, 4},{1, 4}, {1, 2, 4}, {3,4},{4}}.
It is clear X is F_1 space since every g-closed set is F^*-closed set but not $T_{\frac{1}{2}}$ space since {2, 4} is g-closed set but not closed set.

3.20 Lemma

If $A \subseteq Y \subseteq X$ such that Y is closed in X, A is g- closed set in Y then A is g- closed set in X.

3.21 Theorem

Let X be a F_1 topology space. If Y is an open and closed subspace then Y is F_1 space.
Proof.
Assume that A is g-closed set in Y
Since Y is a closed subspace, implies A is g-closed in X [**Lemma (3.20)**].
But X is F_1 space. Then A is F^*- closed set in X
Since Y is an open subspace in X
Thus A is F^*- closed in Y [**Theorem (3.13)**].
Then Y is F_1 space.

3.22 Remark

F_1 space does not hereditary property.

3.23 Theorem

If (X, τ) be F_1 space then every singleton is either F^*- closed or F^*- open.
Proof.
Let X be F_1 space
Assume that $x \in X$ and $\{x\}$ is not F^*- closed set
Since X is the only open set contain $\{x\}^c$ then $\overline{\{x\}^c} \subseteq X$
Hence $\{x\}^c$ g-closed set.
Since X is F_1 space then $\{x\}^c$ is F^*- closed set.
Therefore, $\{x\}$ is F^*- open.

3.24 Definition

A topological space X is called F_2 space if and only if every f-open set in X is open set.

3.25 Example

Let $X = \{1, 2, 3, 4\}$, $\tau = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ be a topology defined on X.
The f-open sets are $\{\emptyset, X, \{1\}, \{2\}, \{1,2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$.
It is clear every f-open set in X is open set.
Then X is F_2 space.

3.26 Proposition

Every F_2 space is F_1 space.
Proof.
Assume that X is F_2 space

Then every f-open set is open set .
Therefore, every g-closed set is F^*- closed set [**Remark (3.9)**].
Thus X is F_1 space.

3.27 Remark

Every F_1 space is F_2 if $\tau = f$.

3.28 Theorem

If (X, τ) be F_2 space then every f-closed set is f-open set.
Proof.
Assume that X is F_2 space
Then $\tau = f$
Therefore, f-open set = f-closed set [**Lemma (3.10)**].
Hence every f-closed set is f-open set.

3.29 Theorem

Let X be F_2 topological space, if Y is an open subspace then Y is F_2 space.
Proof.
Assume that A is f-open set in Y
Since Y is an open subspace, implies A is f-open set in X
But X is F_2 space, then A is open set in X
Therefore, A is open set in Y
Then Y is F_2 space.

3.30 Theorem

If Y is F_2 space and $f : X \longrightarrow Y$ be continuous function, open and surjective, then X is F_2 space.
Proof.
Assume that B is f-open set in X
Since f is continuous, open and surjective, implies $f(B)$ is f-open set in Y
But Y is F_2 space, then (B) is open set in Y
Since f is continuous, then $f^{-1}(f(B))$ is open set in X.
But f is surjective, then $f^{-1}(f(B)) = B$
Therefore, X is F_2 space.

4 Conclusion

In this work, several properties of F_1 space and F_2 space were studied, and these properties, a relationship was drawn between $T_{1/2}$ space and F_1space, there is also relationship between F_1 space and F_2 space.

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Article submitted 21 June 2022. Published as resubmitted by the authors 2 August 2022.

Some Weak Hereditary Properties

https://doi.org/10.31185/wjcm.Vol1.Iss2.33

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Abstract— called f-normal space, which we studied and identified some of its properties as well as relationships with other sets, and we obtained some results that show the relationship between sets using theories obtained using the set from Style (f-open).

Keywords—Normal space and Og-normal space and f-normal space and ff-normal space and f-fg-normal space

1 Introduction

In this chapter we are going to study other features for normal space:

Og-normal space and f-normal space and ff-normal space and f-fg-normal space.

As we know before in general way that said about a topological feature is hereditary, if and only if achieved for each subspace from a space had done. And said about a topological feature is weak hereditary if and only if achieved for each close subspace from a space had done.[1].

Now in particular, we asked the following question:-

Let Ӿ be a topological space, and possesses any of the normal traits above and Y was subset from Ӿ, does sub space Y have the same feature that Ӿ had ?

That is what we are going to justify throughout our study for features of the subset which clarified for each kinds of normal space as state above.

A the beginning, we mention the following theorem which justify if Ӿ was normal space and Y was closed subset if Ӿ then subspace Y is normal space .

2 Preliminaries

2.1 Definition

1. Assume that Ӿ is a topological space and $A \subseteq$ Ӿ. The letter $\overline{A}$ denotes the closur of A is defined by :- $\overline{A} = \cap \{F \subseteq$ Ӿ$; F$ is closed set and $A \subseteq F\}$

2. Let Ꭓ is a topological space and A⊆Ꭓ. The letter A^∘ denotes the interior of A is defined by:-
   A^∘=∪{G⊆Ꭓ:G is open set and G⊆A}.
3. A subset A of a topological space Ꭓ, is called semi-open (s-open)set if there exists an open set O such that O⊆A⊆¯O.
4. A subset A of a topological space Ꭓ is called semi-closed (s-closed) set if there exists a closed set O such that O^∘⊆A⊆O.
5. A subset A of a topological space X is said to be feebly open set if there exists an open set U in X, such that U ⊆ A ⊆ ¯U^s, and the complement of feebly open set is called feebly closed set.
6. Let Ꭓ a topological space, Ꭓ is said to be f-normal space for each two disjoint closed set A and B in Ꭓ there exists are two disjoint f-open set , U and V in Ꭓ such that A⊆U ,B⊆V

2.2 Theorem [2]

Let Ꭓ be normal space and Y closed subset in Ꭓ, then subspace Y is normal space.

2.3 Remark

Look at [3] which justify in the example if Ꭓ is normal space and Y is sub set in Ꭓ, is not necessary sub space Y normal space. Which can be said here the description of normality is not genetic description, in another word its weak genetic description.

2.4 Remark

If Ꭓis og-normal space and Y is subset from Ꭓ, then subspace Y is not necessary og-normal space, as showed that in the following example (2.5). if Y is g-closed subset in Ꭓ then subspace Y be og-normal space.

2.5 Example

Let $Ꭓ = \{a, b, c, d\}$ and
$T_x = \{x, \emptyset, \{a, c\}, \{a, d\}, \{a, c, d\}, \{a\}\}$ a topological space in Ꭓ, and let
$Y = \{a, c, d\} \subset Ꭓ$
And $T_Y = \{\emptyset, Y, \{a, c\}, \{a, d\}, \{a\}$ a topological space in Y
To proof Ꭓ is og-normal space g-closed set in $Ꭓ = \{Ꭓ, \emptyset, \{b\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, a, c, d\}, \{b, c, d\}\}$.
Let $A = Ꭓ$ and $B = \emptyset \Longrightarrow A \cap B = \emptyset$
Let $U = Ꭓ\, and\ V = \emptyset \Longrightarrow U \cap V = \emptyset$
Thus $A \subseteq U$ and $B \subseteq V$
Hence Ꭓ is og-normal space
To proof Y is og-normal space
g-closed set in $Y = \{\emptyset, Y, \{d\}, \{c\}, \{c, d\}\}$

Let $A = \{d\}$ and $B = \{c\} \implies A \cap B = \emptyset$ (g-closed set in Y)
There is not exists two disjoint open set counting {d} and {c}, in T_Y.
Hence Y is not og-normal space
And now we are introducing the following lemma which we need it to proof the coming theorem.

2.6 Lemma

Let Ҳ be a topological space, if A⊆Y⊆Ҳ was and A g-closed set in Y was, and Y g-closed set in Ҳ, then A is going to be g-closed set in Ҳ Proof : Look [4].

We are going to introduce the following theorem which justify if Ҳ og-normal spac was, and Y g-closed subset was in Ҳ then subspace Y is going to be og-normal space:-

2.7 Theorem

Let Ҳ be og-normal space and Y is g-closed subset in Ҳ, then subspace Y is og-normal space. Proof: Let Ҳ is og-normal space, And let A and B are two g-closed set in Y such that A∩B=∅
Hence, by lemma (3.2.5)
A and B a two disjoint g-closed set in Ҳ
Since Ҳ is og-normal space
Hence, there exists two disjoint open set U and V in Ҳ
Such that A⊆U and B⊆V
Let U_1=Y∩U and V_1=Y∩V
Hence U_1 and V_1 are two disjoint open set in Y.(relative topology).
Such that A⊆U_1 and B⊆V_1
Hence, a subspace Y is og-normal space.
Throughout the theorem (3.2.6) we can get the following corollary:-

2.8 Corollary

Let Ҳ be og-normal space and Y is closed subset in Ҳ then subspace Y is og-normal space.

2.9 Remark

Let Ҳ be f-normal space and Y was subset in Ҳ, then subspace is not necessary f normal space, as justify in the following example (3.2.9). and if Y was closed and open set at the same time (clopen) in Ҳ, then subspace Y be f-normal space. We will clarify that in a later theorem

2.10 Example

Let Ӽ $= \{a, b, c, d\}$ and
$T_x = \{x, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}, \{a, b, c\}\}$ a topological space in Ӽ.
See (2.4)
Hence Ӽ is f-normal space,
Now teak $Y = \{a, c, d\} \subset$ Ӽ
$T_Y = \{Y, \emptyset, \{a\}, \{a, d\}, \{a, c\}\}$ a topological space in Y
To proof Y is f-normal space.
f-open set in Y=$\{Y, \emptyset, \{a\}, \{a, c\}, \{a, d\}\}$
now clearly $A\ and\ B$ are two closed set disjoint
let $A = \{c\}$ and $B = \{d\}$
there is not exists two disjoint f-open set counting {c} and {d}, in T_Y.
Hence, subspace Y is not f-normal space.
The following theorem justify if Ӽ is f-normal space and Y is subset closed and open
set (clopen) at the same time in Ӽ then subspace Y is f-normal space.

2.11 Theorem

Let Ӽ be f-normal space and Y is closed and open subset at the same time in Ӽ, then subspace Y is f-normal space according to that subspace Y is f-normal space.

Let Ӽ is f-normal space And let A and B are two closed set in Y such that A∩B=∅
Hence, A and B are two disjoint closed set in Ӽ, by [2].
Since Ӽ is f-normal space
Thus there exists two disjoint f-open set U and V in Ӽ
Such that $A \subseteq U$ and $B \subseteq V$
And let $U_1 = Y \cap U$ and $V_1 = Y \cap V$
Thus, by proposition (1.1.12).
U_1 and V_1 are two disjoint f-open set in Ӽ.
Such that $A \subseteq U_1$ and $B \subseteq V_1$
According to that subspace Y is f-normal space.

2.12 Remark

ff-normal space has the same feature as f-normal space, if Y is was any subset from Ӽ then subspace Y does not ff-normal space. If Y was closed and open subset at the same time in Ӽ, then subspace Y is ff-normal space.

That we going to justify in a later theorem.

2.13 Example

From example(2.19) paragraph (2)
Clearly, Ӽ is ff-normal space
Now, teak $Y = \{a, c, d\} \subset$ Ӽ

$T_Y = \{Y, \emptyset, \{a\}, \{a, d\}, \{a, c\}\}$ a topological space in Y
f-open set in $Y = \{Y, \emptyset, \{a\}, \{a, d\}, \{a, c\}\}$
f-closed set in $Y = \{\emptyset, Y, \{c, d\}, \{c\}, \{d\}\}$
Now notice that $A = \{c\}$ and $B = \{d\}$ are two disjoint f-closed set in Y
there is not exists two disjoint f-open set counting {c} and {d}, both of them in a row.
According to that subspace Y is not ff-normal space.
Now, we are introducing the following lemma which we need it to proof in a later theorem.

2.14 Lemma

Let Ӿ be a topological space, if $A \subseteq Y \subseteq$ Ӿ and A was f-closed set in Y and Y is closed and open set at the same time in Ӿ then A is g-closed set in Ӿ.
Proof:
Assume that Y is closed and open(clopen) subset at the same time in Ӿ,
Let A f-closed set in Y
To proof A is f-closed set in Ӿ.
Since A is f-closed set in Y
Hence, there exist B closed set in Y
Such that $(B^{\circ})_Y \subseteq A \subseteq B$
Since $B^{\circ} = (B^{\circ})_Y \cap Y^{\circ}$ by [5].
Therefore $B^{\circ} = (B^{\circ})_Y \cap Y$ (Y is open set in Ӿ),
Thus $B^{\circ} = (B^{\circ})_Y$
Since B is closed set in Y and Y is closed set in Ӿ,
Hence B is closed set in Ӿ. By [2]
Then $B^{\circ} \subseteq A \subseteq B$
Hence A is f-closed set in Ӿ……..
The following theorem justify that if Ӿ was ff-normal space and Y was open and close
subset at the same time in Ӿ then subspace Y be ff-normal space.

2.15 Theorem

Let Ӿ ff-normal space and Y was open and close subset at the same time in Ӿ then subspace Y is ff-normal space.
Proof:
Assume that Ӿ is ff-normal space
Let it be A and B are two f-closed set in Y. such that $A \cap B = \emptyset$
Hence, by lemma (2.14).
A and B are two disjoint f-closed set in Ӿ
Since Ӿ is ff-normal space
Hence there exists are two disjoint f-open set U and V in Ӿ
Such that $A \subseteq U$ and $B \subseteq V$
Let $U_1 = Y \cap U$ and $V_1 = Y \cap V$

Thus, by proposition (1.1.12)
U_1 and V_1 are two disjoint f-open set in Y
Such that $A \subseteq U_1$ *and* $B \subseteq V_1$
Then, subspace Y is ff-normal space ………

2.16 Remark

Let Ӽ be f-fg-normal space and Y is subset from Ӽ then subspace Y is not necessary be f-fg-normal space at state in coming example (2.17). But if Y is fg-open and close set at the same time in Ӽ then subspace Y is f-fg-normal space. And we will justify that in a later theorem.

2.17

From example (2.39) paragraph (2).
Clear Ӽ is f-fg-normal space
Now teak $Y = \{a, c, d\} \subset$ Ӽ
It is also clear that Y subspace is not f-fg-normal space.
Because $\{c\}$ and $\{d\}$ are two disjoint fg-closed set, But, is not there exists are two disjoint f-open set counting {c} and {d}, both of them in a row. And now we are introducing lemma which we need it to prove the proof of the coming theorem.

2.18 Lemma

Let Ӽ be a topological space, and let $B \subseteq Y \subseteq$ Ӽ such that fg-closed and open set at the same time in Ӽ then B is fg-closed set in Y if and only if B is fg-closed set in Ӽ.
Proof: See [6].

2.19 Theorem

let Ӽ be f-fg-normal space and Y was fg-closed and open set at the same time in Ӽ, then subspace Y is f-fg-normal space.
Proof:
Assume that Ӽ is f-fg-normal space
Let A and B are two fg-closed set in Y,
Such that $A \cap B = \emptyset$
Thus, by lemma (2.18)
A and B are two disjoint fg-closed set in Ӽ
Since Ӽ is f-fg-normal space
Hence there exists are two disjoint f-open set U and V in Ӽ.
Such that $A \subseteq U$ and $B \subseteq V$
Let $U_1 = Y \cap U$ and $V_1 = Y \cap V$
Thus by proposition (1.1.12)
U_1 *and* V_1 are two disjoint f-open set in Y

Such that $A \subseteq U_1$ and $B \subseteq V_1$
Then subspace Y is f-fg-normal space
Through theorem (2.19), the following results can be obtained

3 Conclusion

1. Let Ҳ be f-fg-normal space and Y is f-closed and open subset at the same time then subspace Y is f-fg-normal space.
2. Let Ҳ be f-fg-normal space and Y is closed and open subset at the same time in Ҳ then subspace Y is f-fg-normal space.

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Article submitted 16 October 2021. Published as resubmitted by the authors 29 November 2021.

On Soft Pre-Compact Maps

https://doi.org/10.31185/wjcm.Vol1.Iss2.30

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Abstract— This paper holds to establish a soft pre-compact map and to investigate its associations with soft pre-compact maps, almost soft pre-compact maps, A-almost soft compact maps, A*-almost soft compact maps, mildly soft semi-compact maps, M-mildly soft compact maps besides M*-mildly soft compact maps which are utilized from the relations between their spaces under some conditions or without conditions. Consequently, the composition factors of soft pre-compact maps with soft pre-compact maps, almost soft pre-compact maps, and mildly soft pre-compact maps, A-almost soft compact maps, M-mildly soft compact maps are studied based on the previous association between them. Many examples are given to explain the relationships between the topologies and relations of the soft set.

Keywords— soft pre-compact maps, almost soft pre-compact maps, A-almost soft compact maps, A*-almost soft compact maps, mildly soft pre-compact maps, M-mildly soft compact maps, M*-mildly soft compact maps.

1 Introduction

Molodtsov at the end of the twentieth century presented the soft set with indeterminate information [1]. Afterward, Maji et al. [2] demonstrated numerous novel concepts on soft sets for instance equality, subset, and the complement of a soft set. In 2010, Babitha and Sunil gave the concept of a soft set relation and function, and they explained the composition of functions [3]. Shabir and Naz [4] 2011 originated soft topology and demonstrated some features of soft separation axioms. Aygünoğlu and Aygün [5] established the conception of soft compact spaces. Hida [6] is equipped more powerful explanation for soft compact spaces than space as long as in [5]. Al-Shami et. al. [7] studied unprecedented forms of covering features known as almost soft compact.

Kharal and Ahmad [8] characterized soft maps and instituted principal features. Subsequently, Zorlutuna and Çakir [9] investigated the notion of soft continuous maps. In continuation of their work, Addis et. al. in 2022 proposed a new definition for soft maps and investigate their features [8].

The principal intent of this work is to create a soft pre-compact map and to investigate its correlation between soft pre-compact maps, almost soft pre-compact maps, A-almost soft compact maps, A*-almost soft compact maps, mildly soft semi-compact maps, M-mildly soft compact maps besides M*-mildly soft compact maps which are utilized from the relations between their spaces under some conditions or without conditions. Consequently, the composition factors of soft pre-compact maps with soft pre-compact maps, almost soft pre-compact maps, and mildly soft pre-compact maps, A-almost soft compact maps, M-mildly soft compact maps are studied based on the previous association between them. Many examples are given to explain the relationships between the topologies and relations of the soft set.

2 Preliminaries

2.1 Definition [1]: Let $\mathbb{W}$ be an initial universal set, $\mathbb{E}$ be a set of parameters, and let $\mathbb{P}(\mathbb{W})$ to signas long asy the power set. A pair $(\mathbb{F},\mathbb{E})$ *(* $\mathbb{F}_{\mathbb{E}}$ for short) is known as a soft set as long as $\mathbb{F}$ is a map of $\mathbb{E}$ into the set of all subsets of the set $\mathbb{W}$.

2.2 Definition [2]: Let $\mathbb{F}_{\mathbb{E}}$ be a soft set over $\mathbb{W}$. Subsequently:

1. As long as $\mathbb{F}(\mathbb{e}) = \phi$, for all $\mathbb{e} \in \mathbb{E}$, so $\mathbb{F}_{\mathbb{E}}$ is known as a null soft set and we symbolize it by $\widetilde{\emptyset}$.
2. As long as $\mathbb{F}(\mathbb{e}) = \mathbb{W}$, for all $\mathbb{e} \in \mathbb{E}$, so $\mathbb{F}_{\mathbb{E}}$is known as an absolute soft set and we symbolize it by $\widetilde{\mathbb{W}}$.

2.3 Definition [8]: Let S($\mathbb{W}$,$\mathbb{E}$) with S($\mathbb{M}$,$\mathbb{K}$) are families of all soft sets over $\mathbb{W}$ and $\mathbb{M}$, one by one. The map $\boldsymbol{\varphi_{\psi}}$ is known as a soft map from $\mathbb{W}$ to $\mathbb{M}$, indicated by $\boldsymbol{\varphi_{\psi}}$: S($\mathbb{W}$,$\mathbb{E}$) $\rightarrow$ S($\mathbb{M}$,$\mathbb{K}$), where $\boldsymbol{\varphi}$: $\mathbb{W}\rightarrow \mathbb{M}$ and $\boldsymbol{\psi}$:$\mathbb{E}\rightarrow\mathbb{K}$ are two maps.

1. Let $\mathbb{F}_{\mathbb{E}} \in$ S($\mathbb{W}$,$\mathbb{E}$), therefore the image of $\mathbb{F}_{\mathbb{E}}$ under the soft map φ_{ψ} is the soft set over $\mathbb{M}$ indicated by$\varphi_{\psi}\mathbb{F}_{\mathbb{E}}$ and defined by

$$\varphi_{\psi}(\mathbb{F}_{\mathbb{E}})(\mathbb{k})= \begin{cases} \cup_{\mathbb{e}\in\psi^{-1}(\mathbb{k})\cap\mathbb{E}} \ \varphi\big(\mathbb{F}(\mathbb{e})\big), \text{as long as } \psi^{-1}(\mathbb{K}) \cap \mathbb{E} \neq \emptyset; \\ \emptyset \quad , \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad othrewise. \end{cases}$$

) Let $\mathbb{G}_{\mathbb{K}} \in$ S($\mathbb{M}$,$\mathbb{K}$), therefore the pre-image of $\mathbb{G}_{\mathbb{K}}$ under the soft map φ_{ψ} is the soft set over $\mathbb{W}$ indicated by ${\varphi_{\psi}}^{-1}\,\mathbb{G}_{\mathbb{K}}$ and defined by

$${\varphi_{\psi}}^{-1}(\,\mathbb{G}_{\mathbb{K}})(\mathbb{e})= \begin{cases} \varphi^{-1}(\,\mathbb{G}_{\mathcal{K}}(\psi(\mathbb{e}))), \text{as long as } \psi(\mathbb{e})\in\mathbb{K}; \\ \emptyset \quad , \quad \quad \quad \quad \quad \quad \text{otherwise.} \end{cases}$$

The soft map $\varphi_{\psi}$is known as injective, as long as φ and $\psi$are injective. The soft map $\varphi_{\psi}$is known as surjective, as long as φ and ψ are surjective.

2.4 Definition[4]: Let $\mathbb{T}$ is a family of soft sets over $\mathbb{W}$, $\mathbb{E}$ be a set of parameters. So $\mathbb{T}$ is known as a soft topology on $\mathbb{W}$ as long as the subsequent is satisfied:

1. $\tilde{\phi}$ and $\widetilde{\mathbb{W}}$ are in $\mathbb{T}$.
2. the union of any number of soft sets in $\mathbb{T}$ is in $\mathbb{T}$.
3. the intersection of any two soft sets in $\mathbb{T}$ is in $\mathbb{T}$.

The triple $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ is known as a soft topological space ($\mathcal{STS}$ for short) over $\mathbb{W}$. The members of $\mathbb{T}$ are known as the soft open sets in $\mathbb{W}$. A soft set $\mathbb{F}_{\mathbb{E}}$ over $\mathbb{W}$ is known as a soft closed set in $\mathbb{W}$, as long as its relative complement $\mathbb{F}'_{\mathbb{E}}$ belongs to $\mathbb{T}$.

2.5 Definition[4]: Let $\mathbb{F}_{\mathbb{E}}$ be a non-null soft subset of $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ subsequently $\mathbb{T}_{\mathbb{F}}=\{\mathbb{F}_{\mathbb{E}} \cap \mathbb{G}_{\mathbb{E}}, \forall \mathbb{G}_{\mathbb{E}} \in \mathbb{T}\}$ is known as relative $\boldsymbol{\mathcal{STS}}$ on $\mathbb{F}_{\mathbb{E}}$ and $(\mathbb{F}_{\mathbb{E}}, \mathbb{T}_{\mathbb{F}}, \mathbb{E})$ is known as a soft subspace of $(\mathbb{W}, \mathbb{T}, \mathbb{E})$.

2.6 Definition [15]: A soft subset $\mathbb{F}_{\mathbb{E}}$ of $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ is known as soft pre-open as long as $\mathbb{F}_{\mathbb{E}} \tilde{\subseteq} \boldsymbol{int}(\boldsymbol{cl}\mathbb{F}_{\mathbb{E}})$ with its relative complement is known as soft pre-closed.

2.7 Definition [9]: Let $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ be a $\boldsymbol{\mathcal{STS}}$ over $\mathbb{W}$, $\mathbb{G}_{\mathbb{E}}$ be a soft set over $\mathbb{W}$, and $\boldsymbol{x} \in \mathbb{W}$. Subsequently, $\mathbb{G}_{\mathbb{E}}$ is known as a soft neighborhood of $\boldsymbol{x}_{\mathbb{E}}$, as long as there exists a soft open set $\mathbb{F}_{\mathbb{E}}$ such that $\boldsymbol{x}_{\mathbb{E}} \in \mathbb{F}_{\mathbb{E}} \subseteq \mathbb{G}_{\mathbb{E}}$.

2.8 Definition[10]: Let $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ and $(\mathbb{M}, \mathbb{T}', \mathbb{E})$ be two $\boldsymbol{\mathcal{STS}}$, $\boldsymbol{\mathcal{L}} : (\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be a soft map. For each soft neighborhood $\mathbb{G}_{\mathbb{E}}$ of $\boldsymbol{\mathcal{L}}(\boldsymbol{x}_{\mathbb{E}})$, as long as there exists a soft neighborhood $\mathbb{F}_{\mathbb{E}}$ of $\boldsymbol{x}_{\mathbb{E}}$, such that $\boldsymbol{\mathcal{L}}(\mathbb{F}_{\mathbb{E}}) \subseteq \mathbb{G}_{\mathbb{E}}$, subsequently $\boldsymbol{\mathcal{L}}$ is known as a soft continuous map at $\boldsymbol{x}_{\mathbb{E}}$. As long as $\boldsymbol{\mathcal{L}}$ is a soft continuous map for all $\boldsymbol{x}_{\mathbb{E}}$, subsequently, $\boldsymbol{\mathcal{L}}$ is known as a soft continuous map.

2.9 Definition [14] **:**A soft subset $\mathbb{F}_{\mathbb{E}}$ of $\boldsymbol{\mathcal{STS}}$ $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ is said to be:

1. A soft pre-clopen provided that it is soft pre-open and soft pre-closed,
2. A soft pre-dense provided that $cl\mathbb{F}_{\mathbb{E}} = \mathbb{W}$

2.10 Definition [14]:

1. The collection $\{(\mathbb{F}_{\mathbb{E}\ i} : i \in I\}$ of soft pre-open sets is known as a soft pre-open cover of an $\mathcal{STS}$ $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ as long as $\mathbb{W} = \tilde{\cup}_{i \in I} \mathbb{F}_{\mathbb{E}\ i}$.
2. An STS $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ is known as a soft pre-compact space ($\mathcal{SP}$-compact space for short) as long as each soft pre-open cover of $\mathbb{W}$ has a finite sub-cover of $\mathbb{W}$.

2.11 Definition [14]: A $\mathcal{STS}$ $(\mathbb{W}, \mathbb{T}, \mathbb{E})$is known as almost $\mathcal{SP}$-compact space as long as each soft pre-open cover of $\mathbb{W}$ has a finite sub-cover such that the soft pre-closures whose members cover $\mathbb{W}$.

2.12 Definition [14]: An $\mathcal{STS}$ $(\mathbb{W}, \mathbb{T}, \mathbb{E})$is known as mildly $\mathcal{SP}$-compact space as long as each soft pre-clopen cover of $\mathbb{W}$ has a finite soft subcover $\mathbb{W}$.

2.13 **Proposition [14]:** Each $\mathcal{SP}$-compact space is an almost $\mathcal{SP}$-compact**.**

2.14 **Proposition [14]:** Each almost $\mathcal{SP}$-compact space is a mildly $\mathcal{SP}$-compact**.**

2.15 **Theorem :** Consider $(\mathbb{W}, \mathbb{T}, \mathbb{E})$has a soft pre-base consisting of soft pre-clopen sets. Subsequently, $(\mathbb{W}, \mathbb{T}, \mathbb{E})$is $\mathcal{SP}$-compact as long as and only as long as it is mildly $\mathcal{SP}$-compact.

2.16 **Theorem [14]:** As long as $\mathbb{G}_{\mathbb{E}}$ is an $\mathcal{SP}$-compact subset of $\mathbb{W}$ and $\mathbb{F}_{\mathbb{E}}$is a soft pre-closed subset of $\mathbb{W}$ subsequently $\mathbb{G}_{\mathbb{E}} \cap \mathbb{F}_{\mathbb{E}}$ is $\mathcal{SP}$-compact.

2.17 **Theorem [14]:** As long as $\mathbb{G}_{\mathbb{E}}$is an almost (resp. a mildly) $\mathcal{SP}$-compact subset of $\mathbb{W}_{\mathcal{E}}$ and $\mathbb{F}_{\mathbb{E}}$is a soft pre-clopen subset of $\mathbb{W}$, subsequently $\mathbb{G}_{\mathbb{E}} \cap \mathbb{F}_{\mathbb{E}}$is an almost (resp. a mildly) $\mathcal{SP}$-compact.

2.18 **Proposition [11]:** Let $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ be a $\mathcal{STS}$ and $\mathbb{F}_{\mathbb{E}}$ be any soft set over $\mathbb{W}$.β be an open base of $\mathbb{T}$ subsequently $\boldsymbol{\beta}_{\mathbb{F}_{\mathbb{E}}} = \{\mathbb{G}_{\mathbb{E}} \tilde{\cap} \mathbb{F}_{\mathbb{E}}:\mathbb{G}_{\mathbb{E}} \in \beta\}$ is an open base of $\mathbb{T}_{\mathbb{F}_{\mathbb{E}}}$.

2.19 **Definition [14**]: A collection $\boldsymbol{\beta}$ of soft pre-open sets is known as a soft pre-base of $(\mathbb{W}, \mathbb{T}, \mathbb{E})$as long as each soft pre-open subset of $\mathbb{W}$ can be written as a soft union of members of $\boldsymbol{\beta}$.

2.20 **Theorem [13]:** Let $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ be a $\mathcal{STS}$ each open soft set is pre-open soft.

2.21 **Proposition :** Each soft open base is a soft pre-open base.

Proof: Let $(\mathbb{W}, \mathbb{T}, \mathbb{E})$be a $\mathcal{STS}$ and Let βbe a soft open base. thus,$\mathcal{V}$ is a soft open set, $\forall\ \mathcal{V} \in \beta$. Theorem (2. 11) $\mathcal{V}$ is a soft pre-open set, $\forall\ \mathcal{V} \in \beta$.

3 soft $\mathcal{SP}$-compact map

3.1 Definition: let $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ and $(\mathbb{M}, \mathbb{T}', \mathbb{E})$ be two $\mathcal{STS}$ and let, $\mathcal{L} : (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be a soft map. then, $\mathcal{L}$ is called a $\mathcal{SP}$-compact map, if it is a soft surjective continuous map, and if the pre-image of each $\mathcal{SP}$-compact subset of $\mathbb{M}$ is a $\mathcal{SP}$-compact subset of $\mathbb{W}$.

3.2 Example: Let $\mathbb{W} = \mathbb{R}$, $\mathbb{E}=\{0\}$ and $\mathbb{T}=\{\widetilde{\emptyset}, \widetilde{\mathbb{W}}, \mathbb{G}_{\mathbb{E}}\}$ are $\mathcal{STS}$ on $\mathbb{W}$ such that $\mathbb{G}(\mathbf{0}) = (-\mathbf{1},1)$. A map $\mathcal{L} : (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ such that $\mathcal{L}(\mathcal{X}_{e}) = -\mathcal{X}_{e}$, $\forall\, x \in \mathbb{W}$, therefore $\mathcal{L}$ is a $\mathcal{SP}$-compact map.

3.3 Definition: let $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ and $(\mathbb{M}, \mathbb{T}', \mathbb{E})$ be two $\mathcal{STS}$ and let, $\mathcal{L} : (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be a soft map. then, $\mathcal{L}$ is called an almost $\mathcal{SP}$-compact map, if it is a soft surjective continuous map, and if the pre-image of each almost $\mathcal{SP}$-compact subset of $\mathbb{M}$ is an almost $\mathcal{SP}$-compact subset of $\mathbb{W}$.

3.4 Example: Let $\mathbb{W} = \{x, y, z, h\}$, $\mathbb{E} = \{e_1, e_2\}$ and $\mathbb{T} = \{\widetilde{\emptyset}, \widetilde{\mathbb{W}}, \mathbb{F}_{\mathbb{E}}, \mathbb{G}_{\mathbb{E}}, \mathbb{H}_{\mathbb{E}}, \mathbb{D}_{\mathbb{E}}, \mathbb{K}_{\mathbb{E}}, \mathbb{M}_{\mathbb{E}}\}$ where $\mathbb{F}_{\mathbb{E}} = \{(e_1, \{y\}), (e_2, \emptyset)\}$, $\mathbb{G}_{\mathbb{E}} = \{(e_1, \emptyset), (e_2, h)\}$, $\mathbb{D}_{\mathbb{E}} = \{(e_1, \emptyset), (e_2, \{y, z\})\}$, $\mathbb{H}_{\mathbb{E}} = \{(e_1, \emptyset), (e_2, \{y, h\})\}$ $\mathbb{K}_{\mathbb{E}} = \{(e_1, \{y, z, h\}), (e_2, \emptyset)\}$, $\mathbb{M}_{\mathbb{E}} = \{(e_1, \{x, y, h\}), (e_2, \emptyset)\}$.

Define a soft mapping $\mathcal{L} : (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{W}, \mathbb{T}, \mathbb{E})$ **by**

$\mathcal{L}(e_1, \{x\}) = (e_1, \{x\})$, $\mathcal{L}(e_2, \{x\}) = (e_2, \{x\})$, $\mathcal{L}(e_1, \{y\}) = (e_1, \{z\})$, $\mathcal{L}(e_2, \{y\}) = (e_2, \{z\})$, $\mathcal{L}(e_1, \{z\}) = (e_1, \{y\})$, $\mathcal{L}(e_2, \{z\}) = (e_2, \{y\})$, $\mathcal{L}(e_1, \{h\}) = (e_1, \{h\})$, $\mathcal{L}(e_2, \{h\}) = (e_2, \{h\})$. **Then $\mathcal{L}$ is continuous and surjective mapping , Also $\mathcal{L}$ is an almost $\mathcal{SP}$-compact map.**

3.5 Definition: let $(\mathbb{W}, \mathbb{T}, \mathbb{E})$ and $(\mathbb{M}, \mathbb{T}', \mathbb{E})$ be two $\mathcal{STS}$ and let, $\mathcal{L} : (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be a soft map. then, $\mathcal{L}$ is called a mildly $\mathcal{SP}$-compact map, if it is a soft surjective continuous map, and if the pre-image of each mildly $\mathcal{SP}$-compact subset of $\mathbb{M}$ is a mildly $\mathcal{SP}$-compact subset of $\mathbb{W}$.

3.6 Example: Let $\mathbb{W} = \{m, y, z, h, \mathcal{V}\}$, $\mathbb{E} = \{e_1, e_2\}$. Define a mapping $\mathcal{L} : (\mathbb{W}, \mathbb{T}_{\mathbf{dis}}, \mathbb{E}) \longrightarrow (\mathbb{W}, \mathbb{T}_{\mathbf{dis}}, \mathbb{E})$ by $\mathcal{L}(x_{e}) = x_{e}$ for all $x_{e} \in \mathbb{W}$. Then $\mathcal{L}$ is continuous, surjective. Also is a mildly $\mathcal{SP}$-compact map.

3.7 Definition: let $(\mathbb{W},\mathbb{T},\mathbb{E})$and $(\mathbb{M},\mathbb{T}',\mathbb{E})$be two $\boldsymbol{\mathcal{STS}}$ and let, $\boldsymbol{\mathcal{L}}:(\mathbb{W},\mathbb{T},\mathbb{E}) \longrightarrow (\mathbb{M},\mathbb{T}',\mathbb{E})$be a soft map. then, $\boldsymbol{\mathcal{L}}$ is called A-almost $\boldsymbol{\mathcal{SP}}$-compact map, if it is a soft surjective continuous map, and if the pre-image of each almost $\boldsymbol{\mathcal{SP}}$-compact subset of $\mathbb{M}$ is a $\boldsymbol{\mathcal{SP}}$-compact subset of $\mathbb{W}$.

3.8 Definition: let $(\mathbb{W},\mathbb{T},\mathbb{E})$and $(\mathbb{M},\mathbb{T}',\mathbb{E})$be two $\boldsymbol{\mathcal{STS}}$ and let, $\boldsymbol{\mathcal{L}}:(\mathbb{W},\mathbb{T},\mathbb{E}) \longrightarrow (\mathbb{M},\mathbb{T}',\mathbb{E})$be a soft map. then, $\boldsymbol{\mathcal{L}}$ is called A*-almost $\boldsymbol{\mathcal{SP}}$-compact map, if it is a soft surjective continuous map, and if the pre-image of each $\boldsymbol{\mathcal{SP}}$-compact subset of $\mathbb{M}$ is an almost $\boldsymbol{\mathcal{SP}}$-compact subset of $\mathbb{W}$.

3.9 Definitio: let $(\mathbb{W},\mathbb{T},\mathbb{E})$and $(\mathbb{M},\mathbb{T}',\mathbb{E})$be two $\boldsymbol{\mathcal{STS}}$ and let, $\boldsymbol{\mathcal{L}}:(\mathbb{W},\mathbb{T},\mathbb{E}) \longrightarrow (\mathbb{M},\mathbb{T}',\mathbb{E})$be a soft map. then, $\boldsymbol{\mathcal{L}}$ is called M-mildly $\boldsymbol{\mathcal{SP}}$-compact map, if it is a soft surjective continuous map, and if the pre-image of each mildly $\boldsymbol{\mathcal{SP}}$-compact subset of $\mathbb{M}$ is a $\boldsymbol{\mathcal{SP}}$-compact subset of $\mathbb{W}$**.**

3.10 Definition: let $(\mathbb{W},\mathbb{T},\mathbb{E})$and $(\mathbb{M},\mathbb{T}',\mathbb{E})$be two $\boldsymbol{\mathcal{STS}}$ and let, $\boldsymbol{\mathcal{L}}:(\mathbb{W},\mathbb{T},\mathbb{E}) \longrightarrow (\mathbb{M},\mathbb{T}',\mathbb{E})$be a soft map. then, $\boldsymbol{\mathcal{L}}$ is called M*-mildly $\boldsymbol{\mathcal{SP}}$-compact map, if it is a soft surjective continuous map, and if the pre-image of each $\boldsymbol{\mathcal{SP}}$-compact subset of $\mathbb{M}$ is a mildly $\boldsymbol{\mathcal{SP}}$-compact subset of $\mathbb{W}$.

3.11 Theorem: Every A-almost $\boldsymbol{\mathcal{SP}}$-compact map is a $\boldsymbol{\mathcal{SP}}$-compact map**.**

Proof: let $\mathcal{L}:(\mathbb{W},\mathbb{T},\mathbb{E}) \longrightarrow (\mathbb{M},\mathbb{T}',\mathbb{E})$ be an A-almost $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is a $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathbb{G}_{\mathbb{E}}$ is an almost $\mathcal{SP}$-compact set in $\mathbb{M}$ by Proposition 2.13,. Now $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ soft A-almost $\mathcal{SP}$-compact map. Therefore $\mathcal{L}$ is a $\mathcal{SP}$-compact map. ■

3.12 Theorem : Every A-almost $\boldsymbol{\mathcal{SP}}$-compact map is an almost $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: let $\mathcal{L}:(\mathbb{W},\mathbb{T},\mathbb{E}) \longrightarrow (\mathbb{M},\mathbb{T}',\mathbb{E})$ be an A-almost $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is an almost $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be an almost $\mathcal{SP}$-compact set in $\mathbb{M}$. Now $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$. since $\mathcal{L}$ soft A-almost $\mathcal{SP}$-compact map, now $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set by Proposition 2.13, Therefore $\mathcal{L}$ is an almost $\mathcal{SP}$-compact map. ■

3.13 Theorem: Every A-almost $\boldsymbol{\mathcal{SP}}$-compact map is A*-almost $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: let $\mathcal{L}:(\mathbb{W},\mathbb{T},\mathbb{E}) \longrightarrow (\mathbb{M},\mathbb{T}',\mathbb{E})$ be an A-almost $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is A*-almost $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathbb{G}_{\mathbb{E}}$ be an almost $\mathcal{SP}$-compact set in $\mathbb{M}$ by Proposition 2.13. Now $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ soft A-almost $\mathcal{SP}$-compact map $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost soft compact set Proposition 2.13. Therefore $\mathcal{L}$ is A*-almost $\mathcal{SP}$-compact map. ■

3.14 Theorem: Every A-almost $\mathcal{SP}$-compact map is M*-mildly $\mathcal{SP}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an A-almost $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is an M*-mildly $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathbb{G}_{\mathbb{E}}$ be an almost $\mathcal{SP}$-compact set in $\mathbb{M}$ by Proposition 2.13. Now $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ soft A-almost soft compact map $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$- compact set in $\mathbb{W}$ by Proposition 2.13and Proposition 2.14. Therefore $\mathcal{L}$ is M*-mildly $\mathcal{SP}$-compact map.■

3.15 Theorem: Every A*-almost $\mathcal{SP}$-compact map is M*-mildly $\mathcal{SP}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an A*-almost $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is an M*-mildly $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is a soft A*-almost $\mathcal{SP}$-compact map. now $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$ by Proposition 2.14, Therefore $\mathcal{L}$ is an M*-mildly $\mathcal{SP}$-compact map.■

3.16 Theorem: Every M-mildly $\mathcal{SP}$-compact map is a $\mathcal{SP}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an M-mildly $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is a $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a $\mathcal{SP}$-compact set in $\mathbb{M}$. By Proposition 2.13and Proposition 2.14 $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$ $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is soft M-mildly $\mathcal{SP}$-compact map. Therefore $\mathcal{L}$ is a $\mathcal{SP}$-compact map.■

3.17 Theorem: Every M-mildly $\mathcal{SP}$-compact map is an almost $\mathcal{SP}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an M-mildly $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is an almost $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be an almost $\mathcal{SP}$-compact set in $\mathbb{M}$. By Proposition 2.14 $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is a soft M-mildly $\mathcal{SP}$-compact map. by Proposition 2.13 $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set. Therefore $\mathcal{L}$ is an almost $\mathcal{SP}$-compact map.■

3.18 Theorem: Every M-mildly $\mathcal{SP}$-compact map is a mildly $\mathcal{SP}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an M-mildly $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is a mildly $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is a soft M-mildly $\mathcal{SP}$- compact map. by Proposition 2.13and Proposition 2.14, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$-compact set. Therefore $\mathcal{L}$ is a mildly $\mathcal{SP}$-compact map.■

3.19 Theorem : Every M-mildly $\mathcal{SP}$-compact map is A-almost $\mathcal{SP}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an M-mildly $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is an A-almost $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be an almost $\mathcal{SP}$-compact set in $\mathbb{M}$. By Proposition 2.14 $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$

since $\mathcal{L}$ is a soft M-mildly soft compact map. Therefore $\mathcal{L}$ is an A-almost $\mathcal{SP}$-compact map.■

3.20 Theorem : Every M-mildly $\boldsymbol{\mathcal{SP}}$-compact map is A*-almost $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an M-mildly $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is an A*-almost $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a $\mathcal{SP}$-compact set in $\mathbb{M}$. By Proposition 2.13and Proposition 2.14 $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is a soft M-mildly $\mathcal{SP}$-compact map. By Proposition 2.13 $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore $\mathcal{L}$ is an A*-almost $\mathcal{SP}$-compact map.■

3.21 Theorem: Every M-mildly $\boldsymbol{\mathcal{SP}}$-compact map is an M*-mildly $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an M-mildly $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is an M*-mildly $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a $\mathcal{SP}$-compact set in $\mathbb{M}$ .By Proposition 2.13and Proposition 2.14 $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is a soft M-mildly soft compact map. By Proposition 2.13and Proposition 2.14, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore $\mathcal{L}$ is an M*-mildly $\mathcal{SP}$-compact map.■

3.22 Theorem: Every a mildly $\boldsymbol{\mathcal{SP}}$-compact map is an M*-mildly $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be a mildly $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is an M*-mildly $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a $\mathcal{SP}$-compact set in $\mathbb{M}$ .By Proposition 2.13and Proposition 2.14 $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is a mildly $\mathcal{SP}$-compact map. Therefore $\mathcal{L}$ is an M*-mildly $\mathcal{SP}$-compact map.■

3.23 Theorem : Every an almost $\boldsymbol{\mathcal{SP}}$-compact map is an A*-almost $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an almost $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is an A*-almost $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a $\mathcal{SP}$-compact set in $\mathbb{M}$ .By Proposition 2.13 $\mathbb{G}_{\mathbb{E}}$ is an almost $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is an almost $\mathcal{SP}$-compact map. Therefore $\mathcal{L}$ is an A*-almost $\mathcal{SP}$-compact map.■

3.24 Theorem: Every $\boldsymbol{\mathcal{SP}}$-compact map is A*-almost a $\boldsymbol{\mathcal{SP}}$-compact map**.**

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be a $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is an A*-almost $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-

compact set in $\mathbb{W}$ since $\mathcal{L}$ is a soft compact map. By Proposition 2.13 $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore $\mathcal{L}$ is an A*-almost $\mathcal{SP}$-compact map.■

3.25 Theorem: Every a $\boldsymbol{\mathcal{SP}}$-compact map is an M*-mildly $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be a $\mathcal{SP}$-compact map. T.P $\mathcal{L}$ is an M*-mildly $\mathcal{SP}$-compact map. let $\mathbb{G}_{\mathbb{E}}$ be a $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is a $\mathcal{SP}$-compact map. By Proposition 2.13and Proposition 2.14 $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore $\mathcal{L}$ is an M*-mildly $\mathcal{SP}$-compact map.■

3.26 Theorem: Each $\boldsymbol{\mathcal{SP}}$-compact map is a mildly $\boldsymbol{\mathcal{SP}}$-compact map when the co-domain has a soft pre-base consisting of soft pre-clopen sets.

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be a $\mathcal{SP}$-compact map such that $\mathbb{M}$ has a pre-base consisting of soft pre-clopen sets. Suppose that $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact in $\mathbb{M}$. Since $\mathbb{M}$ has a soft pre-base consisting of soft pre-clopen sets. Subsequently, $\mathbb{G}_{\mathbb{E}}$ has a soft pre-base consisting of soft pre-clopen sets by Theorem 2.21. Thus, $\mathbb{G}_{\mathbb{E}}$is a $\mathcal{SP}$-compact set in $\mathbb{M}$ by Theorem 2.15. So, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ by definition of the $\mathcal{SP}$-compact map. As a result of Proposition 2.13 and proposition 2.14, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\mathcal{L}$is a mildly $\mathcal{SP}$-compact map. ■

3.27 Theorem: Each mildly $\boldsymbol{\mathcal{SP}}$-compact map is a $\boldsymbol{\mathcal{SP}}$-compact map when the domain has a soft pre-base consisting of soft pre-clopen sets.

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be a mildly $\mathcal{SP}$--compact map such that $\mathbb{W}$ has a pre-base consisting of soft pre-clopen sets. Suppose that $\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact set in $\mathbb{M}$. Subsequently, $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$ by Proposition 2.13 and Proposition 2.14, subsequently that $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$ by definition of a mildly$\mathcal{SP}$-compact map. Since $\mathbb{W}$ has a pre-base consisting of soft pre-clopen sets subsequently that $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$has pre-base consisting of soft pre-clopen sets by Theorem 2.21. by Theorem 2.15, that $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$is a $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\mathcal{L}$is a $\mathcal{SP}$-compact map. ■

3.28 Theorem: Each $\boldsymbol{\mathcal{SP}}$-compact map is an almost $\boldsymbol{\mathcal{SP}}$-compact map when the co-domain has a soft pre-base consisting of soft pre-clopen sets.

Proof: Let$\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$be a $\mathcal{SP}$-compact map such that $\mathbb{M}$ has a pre-base consisting of soft pre-clopen sets. Suppose that $\mathbb{G}_{\mathbb{E}}$ is an almost $\mathcal{SP}$-compact set in $\mathbb{M}$, so $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$ by Proposition 2.14. Since $\mathbb{M}$ has a pre-base consisting of soft pre-clopen sets, subsequently $\mathbb{G}_{\mathbb{E}}$ has a soft pre-base consisting of soft pre-clopen sets Theorem 2.21. Thus, $\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact set in $\mathbb{M}$ by Theorem 2.15. Subsequently, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ due to $\mathcal{L}_{\mathcal{E}}$ is a $\mathcal{SP}$-

compact map. Proposition 2.13 implies that $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\mathcal{L}$ is an almost $\mathcal{SP}$-compact map. ■

3.29 Theorem: Each almost $\boldsymbol{\mathcal{SP}}$-compact map is a $\boldsymbol{\mathcal{SP}}$-compact map when the domain has a soft pre-base consisting of soft pre-clopen sets.

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an almost $\mathcal{SP}$-compact map such that $\mathbb{W}$ has a pre-base consisting of soft pre-clopen sets. Let $\mathbb{G}_{\mathbb{E}}$ pre-compact set in $\mathbb{M}$ by Proposition 2.13. $\mathbb{G}_{\mathbb{E}}$is an almost $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$ by defection almost $\mathcal{SP}$-compact map. $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$ by proposition 2.14. $\mathbb{W}$ has a soft pre-base consisting of soft pre-clopen sets subsequently $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ has a soft pre-base consisting of soft pre-clopen sets by Theorem 2.21. As a result of Theorem 2.15 $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$is a $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\mathcal{L}$ is a $\mathcal{SP}$-compact map. ■

3.30 Theorem: Each almost $\boldsymbol{\mathcal{SP}}$-compact map is a mildly $\boldsymbol{\mathcal{SP}}$-compact map When the co-domain has a pre-base of soft pre-clopen sets.

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$be an almost $\mathcal{SP}$-compact map such that $\mathbb{M}$ has a pre-base of a soft pre-clopen set. Suppose that $\mathbb{G}_{\mathbb{E}}$ be a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$. Thus,$\mathbb{G}_{\mathbb{E}}$ has a pre-base of soft pre-clopen sets because $\mathbb{M}$ has a pre-base of soft pre-clopen sets Theorem 2.21. So, $\mathbb{G}_{\mathbb{E}}$is a $\mathcal{SP}$-compact set in $\mathbb{M}$ by Theorem 2.15, and as a result of Proposition 2.13, $\mathbb{M}$ is an almost $\mathcal{SP}$-compact set in $\mathbb{M}$. Thus, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$is an almost $\mathcal{SP}$-compact map. Therefore, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$ by Proposition 2.14 Therefore, $\mathcal{L}$ is a mildly $\mathcal{SP}$-compact map. ■

3.31 Theorem: Each mildly $\boldsymbol{\mathcal{SP}}$-compact map is an almost soft $\boldsymbol{\mathcal{SP}}$-compact map when the domain has a pre-base of soft pre-clopen sets.

Proof: Let $\mathcal{L}: (\mathbb{M}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{W}, \mathbb{T}', \mathbb{E})$ be a mildly $\mathcal{SP}$-compact map such that $\mathbb{W}$ has a pre-base of a soft pre-clopen set. Suppose that $\mathbb{G}_{\mathbb{E}}$is an almost $\mathcal{SP}$-compact set in $\mathbb{M}$. $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$ by Proposition 2.14. Subsequently $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$ by definition of a mildly $\mathcal{SP}$-compact map. $\mathbb{W}$ has a pre-base of soft pre-clopen sets, subsequently, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ has a pre-base of soft pre-clopen sets by Theorem 2.21. As a result of Theorem 2.15, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ by Proposition 2.13,$\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$,Therefore , $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact map.

4 Restriction of type pre-compact maps

4.1 Theorem: let $\boldsymbol{\mathcal{L}}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be a $\boldsymbol{\mathcal{SP}}$-compact map. if $\mathbb{A}_{\mathbb{E}}$ is a pre-closed subset of $\mathbb{W}$ then the restriction $\boldsymbol{g} = \boldsymbol{\mathcal{L}}|(\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E}): (\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ is a $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ is a $\mathcal{SP}$-compact map, $\mathbb{A}_{\mathbb{E}}$ is a pre-closed subset of $\mathbb{W}$, the relative soft topology on $\mathbb{A}_{\mathbb{E}}$ is $\mathbb{T}_{\mathbb{A}} = \{{\mathbb{A}_{\mathbb{E}}}^{*} = \mathbb{A}_{\mathbb{E}} \cap \mathbb{F}_{\mathbb{E}}, \forall \mathbb{F}_{\mathbb{E}} \in \mathbb{T}\}$. Suppose $\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact set in $\mathbb{M}$, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$is a $\mathcal{SP}$-compact set in$\mathbb{W}$ since $\mathcal{L}$ is a $\mathcal{SP}$-compact map. Subsequently, $\mathbb{A}_{\mathbb{E}} \cap \mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}}) \in \mathbb{T}_{\mathbb{A}}$ is a $\mathcal{SP}$-compact set by Theorem2.16. Therefore $g = (\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$is a $\mathcal{SP}$-compact map. ■

4.2 Theorem: Let $\boldsymbol{\mathcal{L}}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$be an almost(resp. a mildly) $\boldsymbol{\mathcal{SP}}$-compact map. If $\mathbb{A}_{\mathbb{E}}$ is a soft pre-clopen subset of $\mathbb{W}$ then the restriction $\boldsymbol{g} = \boldsymbol{\mathcal{L}}|(\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E}): (\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ is an almost(resp. a mildly) $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$be an almost(resp. a mildly) $\mathcal{SP}$-compact map,$\mathbb{A}_{\mathbb{E}}$ is a soft pre-clopen subset of $\mathbb{W}$, the relative topology on $\mathbb{A}_{\mathbb{E}}$ is $\mathbb{T}_{\mathbb{A}} = \{{\mathbb{A}_{\mathbb{E}}}^{*} = \mathbb{A}_{\mathbb{E}} \cap \mathbb{F}_{\mathbb{E}}, \forall \mathbb{F}_{\mathbb{E}} \in \mathbb{T}\}$. Suppose $\mathbb{G}_{\mathbb{E}}$ is an almost(resp. a mildly) $\mathcal{SP}$-compact set in $\mathbb{M}$, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost (resp. a mildly) $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is an almost (resp. a mildly) $\mathcal{SP}$-compact map. Subsequently, $\mathbb{A}_{\mathbb{E}} \cap \mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}}) \in \mathbb{T}_{\mathbb{A}}$ is an almost (resp . a mildly) $\mathcal{SP}$-compact set by Theorem 2.17, Therefore, $g = (\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ is an almost (resp .a mildly) $\mathcal{SP}$-compact map. ■

4.3 Theorem: let $\boldsymbol{\mathcal{L}}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an A-almost(resp. M-mildly) $\boldsymbol{\mathcal{SP}}$-compact map. If $\mathbb{A}_{\mathbb{E}}$ is a soft pre-closed subset of $\mathbb{W}$ then the restriction $\boldsymbol{g} = \boldsymbol{\mathcal{L}}|(\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E}): (\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ is an A-almost(resp. M-mildly) $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$be an A-almost(resp. M-mildly)$\mathcal{SP}$-compact map,$\mathbb{A}_{\mathbb{E}}$ is a soft pre-closed subset of $\mathbb{W}$, the relative topology on $\mathbb{A}_{\mathbb{E}}$ is $\mathbb{T}_{\mathbb{A}} = \{{\mathbb{A}_{\mathbb{E}}}^{*} = \mathbb{A}_{\mathbb{E}} \cap \mathbb{F}_{\mathbb{E}}, \forall \mathbb{F}_{\mathbb{E}} \in \mathbb{T}\}$. Suppose $\mathbb{G}_{\mathbb{E}}$ is an almost(resp. mildly) $\mathcal{SP}$-compact set in $\mathbb{M}$, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is an A-almost (resp. **M-**mildly)$\mathcal{SP}$-compact map. Subsequently, $\mathbb{A}_{\mathbb{E}} \cap \mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}}) \in \mathbb{T}_{\mathbb{A}}$ is a $\mathcal{SP}$-compact set by Theorem2.17. Therefore, $g = (\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ is an A-almost(resp. **M-**mildly)$\mathcal{SP}$-compact map. ■

4.4 Theorem: let $\boldsymbol{\mathcal{L}}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ be an A *-almost(resp. M*-mildly) $\boldsymbol{\mathcal{SP}}$-compact map. If $\mathbb{A}_{\mathbb{E}}$ is a soft pre-clopen subset of $\mathbb{W}$ then the restriction $\boldsymbol{g} = \boldsymbol{\mathcal{L}}|(\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E})$: $(\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ is an A*-almost(resp.M*-mildly) $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \longrightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$be an **A*-**almost(resp.**M*-**mildly) $\mathcal{SP}$-compact map,$\mathbb{A}_{\mathbb{E}}$ is a soft pre-clopen subset of $\mathbb{W}$, the relative topology on $\mathbb{A}_{\mathbb{E}}$ is $\mathbb{T}_{\mathbb{A}} = \{\mathbb{A}_{\mathbb{E}}{}^{*} = \mathbb{A}_{\mathbb{E}} \cap \mathbb{F}_{\mathbb{E}}, \forall \mathbb{F}_{\mathbb{E}} \in \mathbb{T}\}$. Suppose $\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact set in $\mathbb{M}$, $\mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost(resp. mildly)$\mathcal{SP}$-compact set in $\mathbb{W}$ since $\mathcal{L}$ is an A*-almost(resp. **M*-**mildly) $\mathcal{SP}$-compact map. Subsequently, $\mathbb{A}_{\mathbb{E}} \cap \mathcal{L}^{-1}(\mathbb{G}_{\mathbb{E}}) \in \mathbb{T}_{\mathbb{A}}$ is an almost(resp. mildly) $\mathcal{SP}$-compact set by Theorem 2.17. Therefore, $g = (\mathbb{A}_{\mathbb{E}}, \mathbb{T}_{\mathbb{A}}, \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}', \mathbb{E})$ is an A*-almost(resp. **M*-**mildly) $\mathcal{SP}$-compact map. ■

5 Composition of Certain Types of $\boldsymbol{\mathcal{SP}}$-Compact Maps

5.1 Theorem: The composition of $\boldsymbol{\mathcal{SP}}$-compact maps (one by one almost $\boldsymbol{\mathcal{SP}}$-compact maps, mildly $\boldsymbol{\mathcal{SP}}$-compact maps) is also a $\boldsymbol{\mathcal{SP}}$-compact map (one by one almost $\boldsymbol{\mathcal{SP}}$-compact maps, mildly $\boldsymbol{\mathcal{SP}}$-compact map).

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ and $\hbar: (\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ be two $\mathcal{SP}$-compact(one by one almost $\mathcal{SP}$-compact, mildly $\mathcal{SP}$-compact) maps. To veras long asy that $\hbar \circ \mathcal{L}$ is also $\mathcal{SP}$-compact(one by one almost $\mathcal{SP}$-compact, mildly $\mathcal{SP}$-compact) maps. Suppose that $\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact(resp. an almost $\mathcal{SP}$--compact, a mildly $\mathcal{SP}$-compact) set in $\mathbb{M}$. (to show that $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact(one by one an almost $\mathcal{SP}$-compact, a mildly $\mathcal{SP}$-compact) set in $\mathbb{W}$. We have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact(one by one an almost $\mathcal{SP}$-compact, a mildly $\mathcal{SP}$-compact) set in $\mathbb{J}$ since $\hbar$is a $\mathcal{SP}$-compact(one by one an almost $\mathcal{SP}$-compact, a mildly $\mathcal{SP}$-compact) map. Subsequently, $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$is a $\mathcal{SP}$-compact(one by one an almost $\mathcal{SP}$-compact, a mildly $\mathcal{SP}$-compact) set in $\mathbb{W}$ because $\mathcal{L}$ is a $\mathcal{SP}$-compact(one by one an almost $\mathcal{SP}$-compact, a mildly $\mathcal{SP}$-compact) map. We have $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$. so $(\hbar \circ \mathcal{L})^{-1}$is a $\mathcal{SP}$-compact(one by one an almost $\mathcal{SP}$-compact, a mildly $\mathcal{SP}$-compact) set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is also a $\mathcal{SP}$-compact(one by one an almost $\mathcal{SP}$-compact, a mildly $\mathcal{SP}$-compact) map. ■

5.2 Theorem: The composition of A-almost (resp. M-mildly) a $\boldsymbol{\mathcal{SP}}$-compact map is also A-almost (resp. M-mildly) $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ and $\hbar: (\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ be two A-almost (resp. M-mildly) $\mathcal{SP}$-compact maps. To veras long asy that $\hbar \circ \mathcal{L}$ is also A-almost (resp. M-mildly)$\mathcal{SP}$-compact map. Suppose that $\mathbb{G}_{\mathbb{E}}$ is an almost (resp. mildly) $\mathcal{SP}$-compact set in $\mathbb{M}$. (to show that $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact) set in $\mathbb{W}$. We have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{J}$ since $\hbar$ is an A-almost(resp. M-mildly) $\mathcal{SP}$-compact map. By Proposition 2.13 (resp. Proposition 2.13and Proposition 2.14) $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost (resp. mildly) $\mathcal{SP}$-compact set in $\mathbb{J}$. Subsequently,

$\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$is a $\mathcal{SP}$-compact set in $\mathbb{W}$ because $\mathcal{L}$ is an A-almost (resp. M-mildly)$\mathcal{SP}$-compact map. We have $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$ so $(\hbar \circ \mathcal{L})^{-1}$is a $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is an A-almost (resp. M-mildly)$\mathcal{SP}$-compact map. ■

5.3 Theorem: Let $\boldsymbol{\mathcal{L}}$: $(\mathbb{W}, \mathbb{T}, \mathbb{E})\rightarrow(\mathbb{J}, \mathbb{T}', \mathbb{E})$ is a $\boldsymbol{\mathcal{SP}}$-compact map and $\boldsymbol{\hbar}$: $(\mathbb{J}, \mathbb{T}', \mathbb{E})\rightarrow(\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an A-almost(resp. M-mildly) $\boldsymbol{\mathcal{SP}}$-compact map then $\boldsymbol{\hbar \circ \mathcal{L}}$ is an A-almost(resp. M-mildly) $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Let $\mathcal{L}$: $(\mathbb{W}, \mathbb{T}, \mathbb{E})\rightarrow(\mathbb{J}, \mathbb{T}', \mathbb{E})$ be a $\mathcal{SP}$ -compact map and $\hbar$: $(\mathbb{J}, \mathbb{T}', \mathbb{E})\rightarrow(\mathbb{M}, \mathbb{T}'', \mathbb{E})$ A-almost(resp. M-mildly) $\mathcal{SP}$-compact map. To versa long asy that $\hbar \circ \mathcal{L}$ is also A-almost(resp. M-mildly) $\mathcal{SP}$-compact map. Suppose that $\mathbb{G}_{\mathbb{E}}$ is an almost(resp. mildly)$\mathcal{SP}$-compact set in $\mathbb{M}$. (to show that $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact) set in $\mathbb{W}$. We have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{J}$ since $\hbar$ is an A-almost(resp. M-mildly) $\mathcal{SP}$ -compact map. Subsequently, $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$is a $\mathcal{SP}$-compact set in $\mathbb{W}$ because $\mathcal{L}$ is a $\mathcal{SP}$-compact map. We have $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$ so $(\hbar \circ \mathcal{L})^{-1}$is a $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is an A-almost(resp. M-mildly) $\mathcal{SP}$-compact map. ■

5.4 Theorem: Let $\boldsymbol{\mathcal{L}}$: $(\mathbb{W}, \mathbb{T}, \mathbb{E})\rightarrow(\mathbb{J}, \mathbb{T}', \mathbb{E})$ is an A-almost(resp. M-mildly) $\boldsymbol{\mathcal{SP}}$-compact map and $\boldsymbol{\hbar}$:$(\mathbb{J}, \mathbb{T}', \mathbb{E})\rightarrow(\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is a $\boldsymbol{\mathcal{SP}}$-compact map. $\boldsymbol{\hbar \circ \mathcal{L}}$ is a $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: By Theorem 5.1 and Theorem 3.11(resp. by Theorem 5.1 and Theorem 3.16) ■

5.5 Theorem: Let $\boldsymbol{\mathcal{L}}$: $(\mathbb{W}, \mathbb{T}, \mathbb{E})\rightarrow(\mathbb{J}, \mathbb{T}', \mathbb{E})$ is an A-almost (resp. M-mildly) $\boldsymbol{\mathcal{SP}}$-compact map and $\boldsymbol{\hbar}$:$(\mathbb{J}, \mathbb{T}', \mathbb{E})\rightarrow(\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an almost (resp. mildly) $\boldsymbol{\mathcal{SP}}$-compact map. $\boldsymbol{\hbar \circ \mathcal{L}}$ is an A-almost (resp.M-mildly) $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Let $\mathcal{L}$: $(\mathbb{W}, \mathbb{T}, \mathbb{E})\rightarrow(\mathbb{J}, \mathbb{T}', \mathbb{E})$ be an A-almost (M-mildly) $\mathcal{SP}$-compact map and $\hbar$: $(\mathbb{J}, \mathbb{T}', \mathbb{E})\rightarrow(\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an almost(resp. mildly) $\mathcal{SP}$-compact map. To versa long asy that $\hbar \circ \mathcal{L}$ is also A-almost(resp. M-mildly) $\mathcal{SP}$-compact map. Suppose that $\mathbb{G}_{\mathbb{E}}$ is an almost (resp. mildly) $\mathcal{SP}$-compact set in $\mathbb{M}$. (to show that $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact) set in $\mathbb{W}$. We have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost(resp. mildly) $\mathcal{SP}$-compact set in $\mathbb{J}$ since $\hbar$ is an almost (resp. mildly)$\mathcal{SP}$-compact map. Subsequently, $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$is a $\mathcal{SP}$-compact set in $\mathbb{W}$ because $\mathcal{L}$ is an A-almost(resp. M-mildly) $\mathcal{SP}$-compact map. We have $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$ so $(\hbar \circ \mathcal{L})^{-1}$is a $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is an A-almost(resp. M-mildly) $\mathcal{SP}$-compact map. ■

5.6 Theorem: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ is an almost (resp. mildly) $\mathcal{SP}$-compact map and $\hbar :(\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an A-almost (resp. M-mildly)$\mathcal{SP}$-compact map. $\hbar \circ \mathcal{L}$ is an almost(resp. mildly) $\mathcal{SP}$-compact map.

Proof: By Theorem 5.1 and Theorem 3.12(resp. By Theorem 5.1 and Theorem 3.18). ■

5.7 Theorem: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ is an A-almost (resp. M-mildly) $\mathcal{SP}$-compact map and $\hbar :(\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an A*-almost(resp. M*-mildly) $\mathcal{SP}$-compact map then $\hbar \circ \mathcal{L}$ is a $\mathcal{SP}$-compact map.

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ is an A-almost (resp. M-mildly) $\mathcal{SP}$-compact map and $\hbar: (\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an A*-almost(resp. M*-mildly) $\mathcal{SP}$-compact map. To versa long asy that $\hbar \circ \mathcal{L}$ is a $\mathcal{SP}$-compact map. Suppose that $\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact set in $\mathbb{M}$. (to show that $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact) set in $\mathbb{W}$. We have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost(resp. mildly) $\mathcal{SP}$-compact set in $\mathbb{J}$ since $\hbar$ is an A*-almost (resp. M*-mildly) $\mathcal{SP}$-compact map. Subsequently, $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$is a $\mathcal{SP}$-compact set in $\mathbb{W}$ because $\mathcal{L}$ is an A-almost (resp. M-mildly) $\mathcal{SP}$-compact map. We have $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$ so $(\hbar \circ \mathcal{L})^{-1}$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is a $\mathcal{SP}$-compact map. ■

5.8 Theorem: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ is an A*-almost(resp. M*-mildly) $\mathcal{SP}$-compact map and $\hbar :(\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an A-almost(resp. M-mildly) $\mathcal{SP}$-compact map then $\hbar \circ \mathcal{L}$ is an almost (resp. mildly) $\mathcal{SP}$-compact map.

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ is an A*-almost(resp. M*-mildly) $\mathcal{SP}$-compact map and $\hbar: (\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an A-almost(resp. M-mildly) $\mathcal{SP}$-compact map. To versa long asy that $\hbar \circ \mathcal{L}$ is an almost (resp. mildly)$\mathcal{SP}$-compact map. Suppose that $\mathbb{G}_{\mathbb{E}}$ is an almost(resp. mildly) $\mathcal{SP}$-compact set in $\mathbb{M}$. to show that $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is an almost(resp. mildly) $\mathcal{SP}$-compact set in $\mathbb{W}$. We have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is $\mathcal{SP}$-compact set in $\mathbb{J}$ since $\hbar$ is an A-almost(resp. M-mildly) $\mathcal{SP}$-compact map. Subsequently, $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$is an almost(resp. mildly) soft pre-compact set in $\mathbb{W}$ because $\mathcal{L}$ is an A*-almost(resp. M*-mildly) $\mathcal{SP}$-compact map. We have $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$ so $(\hbar \circ \mathcal{L})^{-1}$ is an almost(resp. mildly) $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is an almost(resp. mildly) $\mathcal{SP}$-compact map. ■

5.9 Theorem: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ is an A-almost $\mathcal{SP}$-compact map and $\hbar :(\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an M-mildly $\mathcal{SP}$-compact map then $\hbar \circ \mathcal{L}$ is an M-mildly $\mathcal{SP}$-compact map.

Proof: Let $\mathcal{L}: (\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ be an A-almost $\mathcal{SP}$-compact map and $\hbar: (\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an M-mildly $\mathcal{SP}$-compact map. To versa long asy that $\hbar \circ \mathcal{L}$ is a mildly $\mathcal{SP}$-compact map. Suppose that $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$. to show that $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$. We have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is $\mathcal{SP}$-compact set in $\mathbb{J}$ since $\hbar$ is an M-mildly. By Proposition 2.13 $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set in $\mathbb{J}$. Subsequently, $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$is a $\mathcal{SP}$-compact set in $\mathbb{W}$ because $\mathcal{L}$ is an A-almost $\mathcal{SP}$-compact map. We have $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$ so $(\hbar \circ \mathcal{L})^{-1}$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is an M-mildly $\mathcal{SP}$-compact map. ■

5.10 Theorem: Let $\boldsymbol{\mathcal{L}}$: $(\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ is an M-mildly $\boldsymbol{\mathcal{SP}}$-compact map and $\boldsymbol{\hbar}$:$(\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an A-almost $\boldsymbol{\mathcal{SP}}$-compact map then $\boldsymbol{\hbar \circ \mathcal{L}}$ is an A-almost $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Let $\mathcal{L}$: $(\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ is an M-mildly $\mathcal{SP}$-compact map and $\hbar$: $(\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is an A-almost $\mathcal{SP}$-compact map. To versa long asy that $\hbar \circ \mathcal{L}$ is an A-almost $\mathcal{SP}$-compact map. Suppose that $\mathbb{G}_{\mathbb{E}}$ is an almost $\mathcal{SP}$-compact set in $\mathbb{M}$. to show that $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$. We have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is $\mathcal{SP}$-compact set in $\mathbb{J}$ since $\hbar$ is an A-almost $\mathcal{SP}$-compact map. By Proposition 2.13and Proposition 2.14 $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{J}$. Subsequently, $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$is a $\mathcal{SP}$-compact set in $\mathbb{W}$ because $\mathcal{L}$ is an M-mildly $\mathcal{SP}$-compact map. We have $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$ so $(\hbar \circ \mathcal{L})^{-1}$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is an A-almost $\mathcal{SP}$-compact map. ■

5.11 **Theorem**: Let $\boldsymbol{\mathcal{L}}$: $(\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ be a $\boldsymbol{\mathcal{SP}}$-compact map and $\boldsymbol{\hbar}$: $(\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is a mildly $\boldsymbol{\mathcal{SP}}$-compact map. As long as $(\mathbb{J}, \mathbb{T}', \mathbb{E})$ has a soft pre-base of soft pre-clopen sets, subsequently ,$\boldsymbol{\hbar \circ \mathcal{L}}$ is a mildly $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Suppose $\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{M}$ (to show that $\hbar \circ \mathcal{L}$ is a mildly $\mathcal{SP}$-compact map). we have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{J}$since $\hbar$ is a mildly $\mathcal{SP}$-compact map. Subsequently, $\mathcal{L}$ is a $\mathcal{SP}$-compact map with a co-domain that has a soft pre-base of soft pre-clopen sets. As a result of Theorem 3.26**.** we get $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$. is a mildly $\mathcal{SP}$-compact set in$\mathbb{W}$, because of $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$. so, $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is also a mildly $\mathcal{SP}$-compact map. ■

5.12 **Theorem**: Let $\boldsymbol{\mathcal{L}}$: $(\mathbb{W}, \mathbb{T}, \mathbb{E}) \rightarrow (\mathbb{J}, \mathbb{T}', \mathbb{E})$ is a mildly $\boldsymbol{\mathcal{SP}}$-compact map and $\boldsymbol{\hbar}$: $(\mathbb{J}, \mathbb{T}', \mathbb{E}) \rightarrow (\mathbb{M}, \mathbb{T}'', \mathbb{E})$ is a $\boldsymbol{\mathcal{SP}}$-compact map. As long as $(\mathbb{W}, \mathbb{T}, \mathbb{E})$has a soft pre-base of a soft pre-clopen set Subsequently $\boldsymbol{\hbar \circ \mathcal{L}}$ is a $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Suppose $\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact set in $\mathbb{M}$. (to show that $\hbar \circ \mathcal{L}$ is a $\mathcal{SP}$-compact map). we have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is a $\mathcal{SP}$-compact set in $\mathbb{J}$ since $\hbar$ is a $\mathcal{SP}$-compact map. Subsequently, $\mathcal{L}$ is a mildly $\mathcal{SP}$-compact map with a domain that has a soft pre-base of a soft pre-clopen set. As a result of Theorem 3.27 $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$. is $\mathcal{SP}$-compact set in $\mathbb{W}$. Because of $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}= \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$. So $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is also a $\mathcal{SP}$-compact map. ■

5.13 Theorem: Authorize $\boldsymbol{\mathcal{L}}$: $(\mathbb{W},\mathbb{T},\mathbb{E}) \rightarrow (\mathbb{J},\mathbb{T}',\mathbb{E})$ be a $\boldsymbol{\mathcal{SP}}$-compact map and $\boldsymbol{\hbar}$ $:(\mathbb{J},\mathbb{T}',\mathbb{E}) \rightarrow (\mathbb{M},\mathbb{T}'',\mathbb{E})$ is an almost $\boldsymbol{\mathcal{SP}}$-compact map. As long as $(\mathbb{J},\mathbb{T}',\mathbb{E})$ has a soft pre-base of soft pre-clopen. Subsequently $\boldsymbol{\hbar} \circ \boldsymbol{\mathcal{L}}$ is an almost $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Suppose $\mathbb{G}_{\mathbb{E}}$ is an almost $\mathcal{SP}$-compact set in $\mathbb{M}$. (to show that $\hbar \circ \mathcal{L}$ is an almost $\mathcal{SP}$-compact map). we have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set in $\mathbb{J}$ since $\hbar$ is an almost $\mathcal{SP}$-compact map. Subsequently, $\mathcal{L}$ is a $\mathcal{SP}$-compact map with a co-domain that has a pre-base soft pre-clopen set. As a result of Theorem 3.28, we get $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$ .is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$. Because $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$. So $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathcal{E}}$is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$is also an almost $\mathcal{SP}$-compact map. ■

5.14 Theorem: Let $\boldsymbol{\mathcal{L}}$: $(\mathbb{W},\mathbb{T},\mathbb{E}) \rightarrow (\mathbb{J},\mathbb{T}',\mathbb{E})$ is an almost $\boldsymbol{\mathcal{SP}}$-compact map and $\boldsymbol{\hbar}$: $(\mathbb{J},\mathbb{T}',\mathbb{E}) \rightarrow (\mathbb{M},\mathbb{T}'',\mathbb{E})$ is a $\boldsymbol{\mathcal{SP}}$-compact map. As long as $(\mathbb{W},\mathbb{T},\mathbb{E})$has a pre-base soft pre-clopen set. Subsequently $\boldsymbol{\hbar} \circ \boldsymbol{\mathcal{L}}$ is a $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Suppose $\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact set in $\mathbb{M}$ (to show that $\hbar \circ \mathcal{L}$is $\mathcal{SP}$-$\mathcal{SP}$-compact map). we have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is $\mathcal{SP}$-compact set in $\mathbb{J}$ since $\hbar$ is a $\mathcal{SP}$-compact map. Subsequently, $\mathcal{L}$ is an almost $\mathcal{SP}$-compact map with a domain that has a pre-base soft pre-clopen set. As a result, to Theorem 3.29**.** we get $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$. is a $\mathcal{SP}$-compact set in $\mathbb{W}$. Because of $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$. So $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is a $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is also a $\mathcal{SP}$-compact map. ■

5.15 Theorem: Let $\boldsymbol{\mathcal{L}}$: $(\mathbb{W},\mathbb{T},\mathbb{E}) \rightarrow (\mathbb{J},\mathbb{T}',\mathbb{E})$ is an almost $\boldsymbol{\mathcal{SP}}$-compact map and $\boldsymbol{\hbar}$: $(\mathbb{J},\mathbb{T}',\mathbb{E}) \rightarrow (\mathbb{M},\mathbb{T}'',\mathbb{E})$ is a mildly $\boldsymbol{\mathcal{SP}}$-compact map. As long as $(\mathbb{J},\mathbb{T}',\mathbb{E})$has a soft pre-base of a soft pre-clopen set. Subsequently $\boldsymbol{\hbar} \circ \boldsymbol{\mathcal{L}}$ is a mildly $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Suppose $\mathbb{G}_{\mathbb{E}}$is a mildly $\mathcal{SP}$-compact set in $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$. (to show that $\hbar \circ \mathcal{L}$ is a mildly $\mathcal{SP}$-compact map). we have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$is a mildly $\mathcal{SP}$-$e-$compact set in $\mathbb{J}$since $\hbar$ is a mildly $\mathcal{SP}$-compact map. Subsequently $\mathcal{L}$ is an almost $\mathcal{SP}$-compact map with a co-domain that has a soft pre-base of a soft pre-clopen set. As a result, of Theorem 3.30 we get $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$. is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$. Because of $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$. So $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$is a mildly $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is a mildly $\mathcal{SP}$-compact map. ■

5.16 Theorem:Let $\boldsymbol{\mathcal{L}}$: $(\mathbb{W},\mathbb{T},\mathbb{E}) \rightarrow (\mathbb{J},\mathbb{T}',\mathbb{E})$is a mildly $\boldsymbol{\mathcal{SP}}$-compact map and $\boldsymbol{\hbar}$: $(\mathbb{J},\mathbb{T}',\mathbb{E}) \rightarrow (\mathbb{M},\mathbb{T}'',\mathbb{E})$ is an almost $\boldsymbol{\mathcal{SP}}$-compact map. As long as $(\mathbb{W},\mathbb{T},\mathbb{E})$has a soft pre-base soft pre-clopen set. Subsequently, $\boldsymbol{\hbar} \circ \boldsymbol{\mathcal{L}}$ is an almost $\boldsymbol{\mathcal{SP}}$-compact map.

Proof: Suppose $\mathbb{G}_{\mathbb{E}}$is an almost $\mathcal{SP}$-$\mathcal{SP}$-compact set in $\mathbb{M}$. (to show that $\hbar \circ \mathcal{L}$ is an almost $\mathcal{SP}$-compact map). we have $\hbar^{-1}(\mathbb{G}_{\mathbb{E}})$ is an almost $\mathcal{SP}$-compact set in $\mathbb{J}_{\mathcal{E}}$since $\hbar_{\mathcal{E}}$ is an almost $\mathcal{SP}$-compact map. Subsequently $\mathcal{L}$ is a mildly $\mathcal{SP}$-compact map with a domain that has a pre-base soft pre-clopen set. From Theorem 3.32 we get $\mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$ is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$. Because of $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}} = \mathcal{L}^{-1}(\hbar^{-1}(\mathbb{G}_{\mathbb{E}}))$. So $(\hbar \circ \mathcal{L})^{-1}\mathbb{G}_{\mathbb{E}}$ is an almost $\mathcal{SP}$-compact set in $\mathbb{W}$. Therefore, $\hbar \circ \mathcal{L}$ is an almost $\mathcal{SP}$-compact map. ■

6 Conclusion

To sum up, we create in this paper a soft pre-compact map and investigate its associations with soft pre-compact maps, almost soft pre-compact maps, A-almost soft compact maps, A*-almost soft compact maps, mildly soft semi-compact maps, M-mildly soft compact maps besides M*-mildly soft compact maps which are utilized from the relations between their spaces under some conditions or without conditions. Moreover, the composition factors of soft pre-compact maps with soft pre-compact maps, almost soft pre-compact maps, and mildly soft pre-compact maps, A-almost soft compact maps, M-mildly soft compact maps are studied based on the previous association between them.

7 References

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Article submitted 16 Jun 2022. Published as resubmitted by the authors 1 August 2022.

Concepts Of Bi-Supra Topological Space Via graph Theory

https://doi.org/10.31185/wjcm.Vol1.Iss2.37

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Abstract—Definition of bi-supra topological space via graph theory was introduced in this study. We also studied some concepts related of bi supra-topological space via graph theory. "At least many theorems were proofed as a characterization and some examples introduced to explain the subject".

Keywords—bi-supra topological, subgraph , closure, interior, boundary, exterior.

1 Introduction

Topology is a branch of pure mathematics [8]. In 2019 Gufran Ali [1] we introduced concept bi-supra topological space. In A graph G is defined as a non-empty set of elements called "vertices" and we symbolize it sometimes by with the family of unordered pairs of vertices set and each element of is called "edge" and we symbolize it sometimes by [5]. In 2020 Aiad and Atef [3 , 7] and Abdu [4] and Mahdi[6] the link between graph theory and topological space as the definition topological graph. In this paper new definition bi-supra topological graph with concept of bi-supra topological by graphing concept.

2 Preliminaries

2.1 Definition [1]: suppose X be a non-blank set. "Let $\mathcal{ST}$ be the set of all semi open subset of X (for short $\mathcal{S}o\ x$ [9] and Let $\mathcal{PT}$ be the set of all pre-open subset of X (for short $\mathcal{P}$o(x))[10], then we say that (X,$\mathcal{ST}$, $\mathcal{PT}$) is a bi-supra topological space". after both of (X,$\mathcal{ST}$) and (X, $\mathcal{PT}$) *are supra topological space*".

2.2 Definition: "A subset A of a topological space (X,τ) is called.
a) a pre-open set if [10] A$\subseteq$ int(cl(A)) and a pre-closed set if cl(int(A)) $\subseteq$ A;
b) a semi-open set [9] if A$\subseteq$ cl(int(A)) and a semi-closed set if

int(cl(A)) $\subseteq$ A"

2.3 Definition [5]: "Let $G(V, E)$ be a graph, we call H is a subgraph from

G if $V(H) \subseteq V(G),\ E(H) \subseteq E(G)$, in this case we would write $H \subseteq G$. The spanning subgraph from a graph G is a subgraph acquired by edge deletions only. A deduced subgraph of a graph G is a subgraph acquired by vertex deletions along with the incident edges".

2.4 Definition [3]: "Let $\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{E})$ be a graph, $\boldsymbol{v} \in \boldsymbol{V}(\boldsymbol{G})$ then we define

the post stage vR is the set of all vertices which is not neighborhood of v. S_G is the collection of (vR) is called subbasis of graph. $B_G = \bigcap_{i=1}^{n} S_{G_i}$ is called bases of graph. Then the union of B_G is form a topology on G and $(V(G),\ \tau_G)$ is called topological graph".

2.5 Definition [3]: "Let $\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{E})$ be a graph, $\boldsymbol{H}$ be a subgraph from G.

Then the graph closure of $V(H)$ has the shape":

$$Cl_G(V(H)) = V(H) \cup \{v \in V(G) : vR \cap V(H) \neq \emptyset\}.$$

2.6 Definition [3]:"Let $G = (V, E)$ be a graph, H be a subgraph from G". Then the graph internal of $V(H)$ had the shape:

$$Int_G(V(H)) = \{v \in V(G) : vR \subseteq V(H)\}.$$

2.7 Definition [2]: "Let $G(V, E)$ be a graph that contains a topological graph $(V(G), \tau_G)$, H be a subgraph of G is called open subgraph if $Int_G(V(H)) = V(H)$. It is called closed subgraph if its complement is open subgraph".

2.8 Definition [2]:"Let $G(V, E)$ be a graph which contain a topological graph $(V(G), \tau_G)$, H be a subgraph of G was named semi-open subgraph if $V(H) \subseteq Cl_G(Int_G(V(H)))$." "The family of all semi-open subgraph from G will be denoted by $SO(V(G))$. The complement of a semi-open subgraph is called a semi-closed subgraph and the family of all semi- closed subgraph from G will be denoted by $SF(V(G))$".

Definition [2]: "Let $G(V\ ,\ E)$ be a graph that contains a topological **9 .2** graph $(V(G)\ ,\tau_G)$, H be a subgraph from $G$is called preopen subgraph if $V(H)\ \subseteq Int_G(Cl_G(V(H)))$. The family of all preopen subgraph from G will be denoted by $PO(V(G))$. The complement of a preopen subgraph is called pre-closed subgraph and the family of all pre-closed subgraph from G will be denoted" by $PF(V(G))$.

3 Construct of bi-supra topological via graph

3.1 Definition: suppose $\boldsymbol{G}(\boldsymbol{V}\ ,\ \boldsymbol{E})$ be a graph. Let $\boldsymbol{\mathcal{ST}_G}$ be the set of all open subgraph subset of G and let $\boldsymbol{\mathcal{PT}_G}$be the set of all open subgraph subset of G. Then we say that $(\boldsymbol{V}(\boldsymbol{G}), \boldsymbol{\mathcal{ST}_G}, \boldsymbol{\mathcal{PT}_G})$ is a bi-supra topological graph. When each of $(\boldsymbol{V}(\boldsymbol{G}), \boldsymbol{\mathcal{ST}_G})$ and $(\boldsymbol{V}(\boldsymbol{G}), \boldsymbol{\mathcal{PT}_G})$ are supra a topological graph.

3.2 Example: We construct a topological space for G then

$v_1R = \{v_3\}\ , v_2R = \{v_3, v_4\}\ , v_3R = \{v_1, v_2\}\ , v_4R = \{v_2\}$.

Then a topology subbase was

$S_G = \{\{v_3\},\ \{v_3, v_4\},\ \{v_1, v_2\},\ \{v_2\}\}$.

The base is

$\beta_G = \{V\ (G),\ \emptyset,\ \{v_2\},\ \{v_3\},\ \{v_3, v_4\}, \{v_1, v_2\}\}$.

Hence, the topological graph on G is

$\tau_G = \{V\ (G)\ , \emptyset\ \ , \{v_2\}\ , \{v_3\}\ , \{v_2, v_3\}\ , \{v_1\ , v_2, v_3\}\ , \{v_1\ , v_2\}\}$
$, \{v_2\ , v_3, v_4\}\ , \{v_3\ , v_4\}\}$.

"$\tau_G^c = \{\emptyset, V(G), \{v_1, v_3, v_4\}, \{v_1, v_2, v_4\}, \{v_1, v_4\}, \{v_1\}, \{v_4\}, \{v_1, v_2\},$
$\{v_3\ , v_4\}\}$.

$\mathcal{ST}_G = \{V(G), \emptyset, \{v_2\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\},$
$\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}$.

$\mathcal{PT}_G = \{V(G), \emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_3\}, \{v_2, v_3\},$
$\{v_2, v_4\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}$".

Hence $(V(G), \mathcal{ST}_G, \mathcal{PT}_G)$ is bi-supra topological graph.

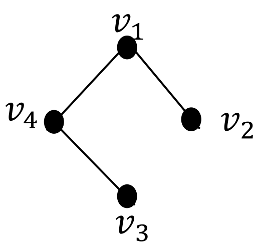

Figure1. Simple graph

3.3 Definition: Let $(\boldsymbol{V(G)}, \boldsymbol{\mathcal{ST}_G}, \boldsymbol{\mathcal{PT}_G})$ be a bi-supra topological graph

and suppose $V(H)$ be a subgraph of $V(G)$. So $V(H)$ was supposed to be:
1- "$(\mathcal{ST}_G, \mathcal{PT}_G)$-supra open subgraph if
$V(H) = V(K) \cup V(L)\ where\ V(K) \in \mathcal{ST}_G\ and\ V(L) \in \mathcal{PT}_G$. The complement of $(\mathcal{ST}_G, \mathcal{PT}_G)$-supra open subgraph is called $(\mathcal{ST}_G, \mathcal{PT}_G)$-supra closed subgraph.
2-$(\mathcal{ST}_G, \mathcal{PT}_G)^*$- supra open subgraph if
$V(H) = V(K) \cup V(L)\ where\ V(K) \in \mathcal{ST}_G\ andV(L) \in \mathcal{PT}_G$ such that $V(L) \notin \mathcal{ST}_G\ or\ V(K) \in \mathcal{PT}_G, V(L) \in \mathcal{ST}_G\ such\ that\ V(K) \notin \mathcal{PT}_G.$
The complement of $(\mathcal{ST}_G, \mathcal{PT}_G)^*$-supra open subgraph is called $(\mathcal{ST}_G, \mathcal{PT}_G)^*$-supra closed subgraph.
3- bi- supra open subgraph if $V(G) = V(K)$ where, $V(K) \in \tau_G$. The complement of bi-supra open subgraph is called bi-closed subgraph".

3.4 Proposition:

1. "Every bi-supra open subgraph is $(\mathcal{ST}_G, \mathcal{PT}_G)$-supra open subgraph and every bi-supra closed subgraph is $(\mathcal{ST}_G, \mathcal{PT}_G)$-supra closed subgraph but the convers is not true.
2. Every $(\mathcal{ST}_G, \mathcal{PT})^*$-supra open subgraph is $(\mathcal{ST}_G, \mathcal{PT}_G)$-supra open graph and Every $(\mathcal{ST}_G, \mathcal{PT})^*$-supra closed subgraph is $(\mathcal{ST}_G, \mathcal{PT}_G)$-supra closed graph but the converse is not true.
3. The $(\mathcal{ST}_G, \mathcal{PT}_G)^*$-supra open subgraph, bi-supra open subgraph are independent and The $(\mathcal{ST}_G, \mathcal{PT}_G)^*$-supra closed subgraph, bi-supra closed subgraph are independent".

3.5 Remark: The set of all $(\boldsymbol{\mathcal{ST}_G}, \boldsymbol{\mathcal{PT}_G})$[res. $(\boldsymbol{\mathcal{ST}_G}, \boldsymbol{\mathcal{PT}_G})^*$, bi]-supra open

subgraph and $(\mathcal{ST}_G, \mathcal{PT}_G)$[res. $(\mathcal{ST}_G, \mathcal{PT}_G)^*$, bi]-supra closed subgraph was require not essentially form a topological graph it was a supra topological graph.

3.6 Example: From Example 3.2,

"$\tau_G = \{V\ (G)\ , \emptyset\ , \{v_2\}\ , \{v_3\}\ , \{v_2, v_3\}\ , \{v_1\ , v_2, v_3\}\ , \{v_1\ , v_2\}\}$

$\mathcal{ST}_\mathrm{G} = \{\mathrm{V(G)}, \emptyset, \{\mathrm{v}_2\}, \{\mathrm{v}_3\}, \{\mathrm{v}_1, \mathrm{v}_2\}, \{\mathrm{v}_1, \mathrm{v}_3\}, \{\mathrm{v}_2, \mathrm{v}_3\}, \{\mathrm{v}_2, \mathrm{v}_4\}, \{\mathrm{v}_3, \mathrm{v}_4\},$

$\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}, \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_4\}, \{\mathrm{v}_1, \mathrm{v}_3, \mathrm{v}_4\}, \{\mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}\}.$

$\mathcal{PT}_\mathrm{G} = \{\mathrm{V(G)}, \emptyset, \{\mathrm{v}_1\}, \{\mathrm{v}_2\}, \{\mathrm{v}_3\}, \{\mathrm{v}_4\}, \{\mathrm{v}_1, \mathrm{v}_2\}, \{\mathrm{v}_1, \mathrm{v}_3\}, \{\mathrm{v}_1, \mathrm{v}_4\}, \{\mathrm{v}_2, \mathrm{v}_3\},$

$\{\mathrm{v}_2, \mathrm{v}_4\}, \{\mathrm{v}_3, \mathrm{v}_4\}, \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}, \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_4\}, \{\mathrm{v}_1, \mathrm{v}_3, \mathrm{v}_4\}, \{\mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}\}.$

"$(\mathcal{ST}_\mathrm{G}, \mathcal{PT}_\mathrm{G})$-open supra subgraph".

$= \{V(G), \emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\},$

$\{v_1, v_3\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\},$

$\{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}.$

$(\mathcal{ST}_G, \mathcal{PT}_G)$-closed supra subgraph $= \{V(G), \emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\},$

$\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_1, v_2, v_3\},$

$\{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}.$

$(\mathcal{ST}_G, \mathcal{PT}_G)^*$-supra open subgraph$= \{V(G), \emptyset, \{v_1\}, \{v_4\}, \{v_1, v_4\}\}$

$(\mathcal{ST}_G, \mathcal{PT}_G)^*$-supra closed subgraph $= \{V(G), \emptyset, \{v_2, v_3, v_4\}, \{v_1, v_2, v_3\},$

$\{v_2, v_3\}\}.$

Bi-supra open subgraph $= \{V(G), \emptyset, \{v_2\}, \{v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\},$

$\{v_1, v_2\}\}.$

Bi-supra closed subgraph $= \{V(G), \emptyset, \{v_1, v_3, v_4\}, \{v_1, v_2, v_4\}, \{v_1, v_4\},$

$\{v_4\}, \{v_3, v_4\}\}$".

4 Some Concepts of Bi-supra topological graph

4.1 Definition: "suppose $\boldsymbol{G(V, E)}$ be a graph, $\boldsymbol{V(H)}$ be a subgraph from $\boldsymbol{V(G)}$. "Then the graph closure of bi-supra topological graph $(\boldsymbol{V(G)}, \boldsymbol{\mathcal{ST}_G}, \boldsymbol{\mathcal{PT}_G})$

have the shape: "$bi - Cl_G(V(H)) = \cap \{V(K): V(H) \subseteq V(K), V(K)\ is\ bi - supra\ \ closed\ subgraph\}$."

4.2 Theorem: "suppose $\boldsymbol{G(V, E)}$ be a graph that contain bi-supra

topological graph $(V(G), \mathcal{ST}_G, \mathcal{PT}_G)$." If H , W are subgraphs from G; then:

(1) $V(H) \subseteq bi - Cl_G(V(H))$.

(2) If $H \subseteq W$, then $bi - Cl_G(V(H)) \subseteq bi - Cl_G(V(W))$.

(3)" $bi - Cl_G(V(H) \cup V(W)) = bi - Cl_G(V(H) \cup bi - Cl_G(V(W))$."

(4)" $bi - Cl_G(V(H) \cap V(W)) \subseteq bi - Cl_G(V(H)) \cap bi - (V(W))$."

Proof : 1- Suppose that $v \in V(H)$, by definition 4.2.Then

$V(H) \in bi - Cl_G(V(H))$. Therefore, $V(H) \subseteq bi - Cl_G(V(H))$

(2) From (1), $V\ (H) \subseteq bi - Cl_G(V(H)),\ V(W) \subseteq bi - Cl_G(V(W))$.

Since, $H \subseteq W$, then $V(H) \subseteq V(W)$. Therefore,

$bi - Cl_G(V(H)) \subseteq bi - Cl_G(V(W))$.

(4) From (1), $V(H) \subseteq bi - Cl_G(V(H)), V(W) \subseteq bi - Cl_G(V(W))$. Since

$V(H) \cap V(W) \subseteq V(H),\ V(H) \cap V(W) \subseteq V(W)$. Then

"$bi - Cl_G(V(H) \cap V(W)) \subseteq bi - Cl_G(V(H))$,

$bi - Cl_G(V(H) \cap V(W)) \subseteq bi - Cl_G(V(W))$. Therefore,

$bi - Cl_G(V(H) \cap V(W)) \subseteq bi - Cl_G(V(H) \cap bi - Cl_G(V(W))$."

4.3 Definition: Let $\boldsymbol{G}(\boldsymbol{V}\ ,\ \boldsymbol{E})$ be a graph, $\boldsymbol{H}$ be a subgraph from $\boldsymbol{G}$.

So the graph internal of bi-supra topological graph $(V(G), \mathcal{ST}_G, \mathcal{PT}_G)$ had the shape:
"$bi - Int_G(V(H)) = \cup \{V(L): V(L) \subseteq V(H), V(L)\ is\ bi - supra\ open\ subgraph\}$."

4.4 Theorem: Suppose $\boldsymbol{G}(\boldsymbol{V}\ ,\ \boldsymbol{E})$ be a graph that contain bi-supra topological graph $(V(G), \mathcal{ST}_G, \mathcal{PT}_G)$. If H , W are subgraphs from G; then:

(1) If $H \subseteq G$, then $bi - Int_G(V(H)) \subseteq V(H)$.
(2) If $H \subseteq W$, then $bi - Int_G(V(H)) \subseteq \text{bi} - Int_G(V(W))$.
(3) $bi - Int_G(V(H) \cap V(W)) = bi - Int_G(V(H)) \cap \text{bi} - Int_G(V(W))$.
(4) $bi - Int_G(V(H)) \cup bi - Int_G(V(W)) \subseteq bi - Int_G(V(H) \cup V(W))$.

Proof: (4) Suppose that, $V(H)$, $V(W) \subseteq V(G)$, since
$V(H) \subseteq V(H) \cup V(W), V\ (W) \subseteq V\ (H) \cup V\ (W)$. Then
$bi - Int_G(V(H)) \subseteq bi - Int_G(V(H) \cup V(W))$,
$bi - Int_G(V(W)) \subseteq bi - Int_G(V(H) \cup V(W))$. Therefore,
$bi - Int_G(V(H)) \cup bi - Int_G(V(W)) \subseteq bi - Int_G(V(H) \cup V(W))$.

4.5 Example: From Example 3.6. Let $\boldsymbol{V}(\boldsymbol{H}) = \{\boldsymbol{v_1}, \boldsymbol{v_3}\}$.

Bi-supra open subgraph=" $\{V\ (G)$, $\emptyset$, $\{v_2\}$, $\{v_3\}$, $\{v_2, v_3\}$, $\{v_1 , v_2, v_3\}$, $\{v_1 , v_2\}\}$."
Bi-supra closed subgraph=" $\{V\ (G)$, $\emptyset$, $\{v_1, v_3, v_4\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_4\}$, $\{v_4\}$, $\{v_3, v_4\}\}$."
Then $bi - Cl_G(V(H)) = \{v_1, v_3, v_4\}$ and
$bi - Int_G(V(H)) = \{v_3\}$.

4.6 **Remark**: Suppose $\boldsymbol{G}(\boldsymbol{V},\ \boldsymbol{E})$ be a graph, $\boldsymbol{H}$ is a subgraph from $\boldsymbol{G}$.

Then, $bi-Int_G(V(H)) \subseteq V(H) \subseteq bi-Cl_G(V(H))$

4.7 **Proposition**: "Let $\boldsymbol{G}(\boldsymbol{V}\ ,\ \boldsymbol{E})$ be a graph that contains bi-supra

topological graph $(V(G), \mathcal{ST}_G, \mathcal{PT}_G)$". If H be a subgraph from G, so:
(1) $bi-Cl_G(V(G)\ -\ V(H))\ =\ V(G)\ -\ bi-Int_G(V(H))$.
(2) $bi-Int_G(V(G)\ -\ V(H))\ =\ V(G)\ -\ bi-Cl_G(V(H))$.

Proof:

(1) Suppose that $v\ \in\ V(G)\ -\ V(H)$, then $v\ \in\ V(G), v\ \notin\ V(H)$. By Theorem 4.4, $v\ \notin\ bi-Int_G(V(H))$.
So, $v\ \in\ V(G)\ -\ bi-Int_G(V(H))$.
Conversely,
Assume that $v\ \in\ V(G)\ -\ bi-Int_G(V(H)), v\ \in\ V(G)$ and by definition 4.3, $v\ \notin\ bi-Int_G(V(H))$, Then,
$v\ \in\ V(G),\ V(K) \subseteq V(H)$, for every
$v\ \in\ V(G)$. So $v\ \in\ V(G),\ v\ \notin\ V(H)$. This means $v\ \in\ V(G)\ -\ V(H)$,
so $V(G)\ -\ V(H)\ \subseteq\ bi-Cl_G(V(G)\ -\ V(H))$. Therefore,
$bi-Cl_G(V(G)\ -\ V(H))\ =\ V(G)\ -bi-\ Int_G(V(H))$.
(2) Assume that $v\ \in\ bi-Int_G(V(G)\ -\ V(H))$. Then by definition 4.3, for every $v\ \in\ V(G)$ such that $V(L)\ \subseteq\ V(G)\ -\ V(H)$. Then
$V(L) \subseteq\ V(G), V(L) \nsubseteq\ V(H)$. This means
$v\ \in\ bi-Int_G(V(G))\ , V(L)\ \cap\ V(H)\ =\ \emptyset$. Then,
$v\ \in\ V(G), v\ \notin\ bi-Cl_G(V(H))$.
Therefore, $v\ \in\ V(G)\ -\ bi-Cl_G(V(H))$.

4.8 **Definition:** "Let $\boldsymbol{G}(\boldsymbol{V},\ \boldsymbol{E})$ be a graph, $\boldsymbol{H}$ be a subgraph from $\boldsymbol{G}$.

Then the graph exterior of bi-supra topological graph has the shape":
$bi-Ext_G(V(H)) = bi-Int_G(C\ (V(H)))$ or
$bi-Ext_G(V(H)) =\ C(bi-Cl_G(V(H)))$

4.9 **Theorem:** "Let $\boldsymbol{G}(\boldsymbol{V}\ ,\ \boldsymbol{E})$ be a graph that contains bi-supra

topological graph $(V(G), \mathcal{ST}_G, \mathcal{PT}_G)$. If $H\ , W$ are subgraphs from G; then":
(1) $bi-Ext_G(V(H))\ \cap\ V(H)\ =\ \emptyset$.
(2) If $H\ \subseteq\ W$, then $bi-Ext_G(V(W))\ \subseteq\ bi-Ext_G(V(H))$.
(3) $bi-Ext_G(V(H) \cup V(W)) \subseteq bi-Ext_G(V(H)) \cap bi-Ext_G(V(W))$.

4.10 **Definition**: Suppose $\boldsymbol{G}(\boldsymbol{V},\ \boldsymbol{E})$ be a graph, $\boldsymbol{H}$ be a sub-graph from $\boldsymbol{G}$.

So, the graph boundary of bi-supra topological graph has the shape:
$bi-B_G(V(H)) =\ bi-Cl_G(V(H)) - bi-Int_G(V(H))$.

4.11 Theorem: Suppose $\boldsymbol{G}(\boldsymbol{V}\ ,\ \boldsymbol{E})$ be a graph that contain bi-supra topological graph $(V(G), \mathcal{ST}_G, \mathcal{PT}_G)$. If H be a sub-graph from G so

(1) $\mathrm{bi}-\mathrm{B_G}(\mathrm{V(H)})\ \cap\ \mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\ =\ \emptyset$.

(2) $\mathrm{bi}-\mathrm{B_G}(\mathrm{V(H)})\ \cap\ \mathrm{bi}-\mathrm{Ext_G}(\mathrm{V(H)})\ =\ \emptyset$.

(3) $\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\ \cap\ \mathrm{bi}-\mathrm{Ext_G}(\mathrm{V(H)})\ =\ \emptyset$.

(4) $\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\ \cup\ \mathrm{bi}-\mathrm{Ext_G}(\mathrm{V(H)})\ \cup\ \mathrm{bi}-\mathrm{B_G}(\mathrm{V(H)})\ =\ \mathrm{V(G)}$.

(5) $\mathrm{Cl_G}(\mathrm{V(H)}\ =\ \mathrm{Int_G}(\mathrm{V(H)})\ \cup\ \mathrm{B_G}(\mathrm{V(H)})$.

Proof:
(1) By definition 4.10, it`s clear.
(2) Assume that $V(H)\ \subseteq\ V(G),\ bi-B_G(V(H))\cap\ bi-Ext_G(V(H))$, by theorem 4.9

$\rightarrow \mathrm{bi}-\mathrm{B_G}(\mathrm{C(V(H))})\ \cap\ \mathrm{bi}-\mathrm{Int_G}(\mathrm{C(V(H))})=\ \emptyset$,

$=\mathrm{bi}-\mathrm{B_G}(\mathrm{G}\ -\ \mathrm{V\,(H)})\ \cap\ \mathrm{bi}-\mathrm{Int_G}(\mathrm{C(G}\ -\ \mathrm{V\,(H)}))\ =\ \emptyset$.

(3) Assume that $\mathrm{V(H)}\ \subseteq\ \mathrm{V(G)},\ \mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\ \cap\ \mathrm{bi}-\mathrm{Ext_G}(\mathrm{V(H)})$

$=\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\cap\mathrm{bi}-\mathrm{Int_G}(\mathrm{C(V(H))})\subseteq\mathrm{V(H)}\cap\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})$,

$=\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\ \cap\ (\mathrm{C(bi}-\mathrm{Cl_G}(\mathrm{V(H)}))$

$=\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\ \cap\ (\mathrm{G}\ -\ \mathrm{bi}-\mathrm{Cl_G}(\mathrm{V(H)}))$, by distributing intersection,

$\rightarrow(\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\ \cap\ \mathrm{G})\ -\ (\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\ \cap\ \mathrm{bi}-\mathrm{Cl_G}(\mathrm{V(H)})$

$=\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\ -\ \mathrm{bi}-\mathrm{Ext_G}(\mathrm{V(H)})\ =\ \emptyset$.

(4) Assume that $\mathrm{V(H)}\ \subseteq\ \mathrm{V(G)}$,

$\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\ \cup\ \mathrm{bi}-\mathrm{Ext_G}(\mathrm{V(H)})\ \cup\ \mathrm{bi}-\mathrm{B_G}(\mathrm{V(H)})$

$=\mathrm{bi}-\mathrm{Cl_G}(\mathrm{V(H)})\ \cup\ \mathrm{C(bi}-\mathrm{Cl_G}(\mathrm{V(H)}))\ =\ \mathrm{G}$.

(5) Suppose $\mathrm{v}\ \in\ \mathrm{V(H)}$,

$\mathrm{bi}-\mathrm{B_G}(\mathrm{V(H)})\ \cup\ \mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})=\ (\mathrm{bi}-\mathrm{Cl_G}(\mathrm{V(H)})\ -$
$\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)}))\ \cup\mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})$. Therefore,

$\mathrm{bi}-\mathrm{Cl_G}(\mathrm{V(H)}\ =\ \mathrm{bi}-\mathrm{Int_G}(\mathrm{V(H)})\ \cup\ \mathrm{bi}-\mathrm{B_G}(\mathrm{V(H)})$.

4.12 Example: From example 3.6, let $\boldsymbol{V}(\boldsymbol{H}) = \{\boldsymbol{v_1}, \boldsymbol{v_3}\}$. Then

$\mathrm{bi} - \mathrm{Cl_G}(\mathrm{V(H)}) = \{\mathrm{v_1}, \mathrm{v_3}, \mathrm{v_4}\}$ and

$\mathrm{bi} - \mathrm{Int_G}(\mathrm{V(H)}) = \{\mathrm{v_3}\}$.

$\mathrm{bi} - \mathrm{Ext_G}(\mathrm{V(H)}) = \{\mathrm{v_2}\}$.

$\mathrm{b}i - B_G(V(H)) = \{v_1, v_4\}$.

Definition**: Suppose $\boldsymbol{G}(\boldsymbol{V}\ ,\ \boldsymbol{E})$ be a graph, $\boldsymbol{H}$ be a sub-graph from** G, and $(V(G), \mathcal{ST}_G, \mathcal{PT}_G)$ bi-supra topological graph is called:

1- $(\mathcal{ST}_G, \mathcal{PT}_G)^* Cl_G(V(H)) = \cap \{V(K): V(H) \subseteq V(K), V(K)\ is\ (\mathcal{ST}_G, \mathcal{PT}_G)^* - closed\ subgraph\}$.

2- $(\mathcal{ST}_G, \mathcal{PT}_G)^* Int_G(V(H)) = \cup \{V(L): V(L) \subseteq V(H), V(L)\ is\ (\mathcal{ST}_G, \mathcal{PT}_G)^* - open\ subgraph\}$.

4.13 Example: From example 3.6, let $\boldsymbol{V}(\boldsymbol{H}) = \{\boldsymbol{v_1}, \boldsymbol{v_3}\}$. Then

$(\mathcal{ST}_G, \mathcal{PT}_G)^* Cl_G(V(H)) = \{v_1, v_2, v_3\}$.

$(\mathcal{ST}_G, \mathcal{PT}_G)^* Int_G(V(H)) = \{v_1\}$.

4.14 Proposition: Suppose $\boldsymbol{G}(\boldsymbol{V},\ \boldsymbol{E})$ be a graph, $\boldsymbol{H}$ is a sub-graph from $\boldsymbol{G}$.

Then, $(\mathcal{ST}_G, \mathcal{PT}_G)^* Int_G(V(H)) \subseteq V(H) \subseteq (\mathcal{ST}_G, \mathcal{PT}_G)^* Cl_G(V(H))$

5 Conclusions

The main result in this paper is to explain the relations between bi-supra Topological Space and topological graph theory which illustrated by many proposition as 3.4 and some examples 3.6 and another theorem by 4.2,4.4 and 4.9 by these theorems can study more of subject in graph theory.

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Article submitted 23 June 2022. Published as resubmitted by the authors 1 August 2022.

Effect of Couple Stress on Peristaltic Transport of Powell-Eyring Fluid Peristaltic flow in Inclined Asymmetric Channel with Porous Medium

https://doi.org/10.31185/wjcm.Vol1.Iss2.31

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Abstract— The goal of this study is to investigate the effect of couple stress on Powell-Eyring fluid peristaltic transport in an inclined asymmetric channel using porous medium. Peristaltic motion of a magnetohydrodynamic Powell-Eyring fluid in inclined asymmetric channel with porous medium, medium is investigated in the present study. The modeling of mathematic is created in the presence of effect of couple stress, using constitutive equations following the Powell-Eyring fluid model. In flow analysis, assumptions such as long wave length approximation and low Reynolds number are utilized. Closed form formulas for the stream function and mechanical efficiency are created. On the channel walls, pressure rise per wave length has been calculated numerically. The effects of the Hartman number (Ha), Darcy number (Da), material fluid parameter (w), inclination of magnetic field (β), amplitude ratio (∅), The effects of the Couple Stress on axial velocity and entrapment are investigated in detail and graphically shown.

Keywords— Couple Stress, Powell-Eyring Fluid, Porous Medium

1 Introduction

The peristaltic motion is a series of contractions and diastoles that push fluid along the path, making it easier to move. Peristalsis is a natural property of smooth muscles and tubes that carry fluid through vessels as a result of motor activity in numerous biological systems, the passage of urine from the kidney to the bladder, the movement of food through the gastrointestinal tract, and the migration of eggs through the fallopian tube are all examples of this movement [1]. Many researchers study peristaltic transport with heat transfer (with or without porous medium) in a range of subjects and applications, including: [2],[3] investigated the combined influence of velocity slip, temperature, and density jump conditions on MHD peristaltic transport of a Carreau fluid in a non-uniform channel [4], to investigate the impact of Heat generation on MHD peristaltic flow in a nanofluid with compliant walls. Pair fluid behavior studies are essential for understanding a range of physical issues, and they can better describe the behavior of rheologic ally complicated fluids including liquid crystals, polymeric suspensions with long chain molecules, lubrication, and human/sub-human blood. A couple-stress fluid is a non-Newtonian fluid with specified particle sizes. In classical continuum theory, the effects of particle sizes are not examined. Peristaltic transmission of couple-stress fluid has been studied recently [5,6,7, 8]. Despite the fact that there is always some slide in real systems, several of the experiments on couple-stress fluids defined above employed blood as a couple stress fluid and were carried out under no slip conditions. Peristalsis is a natural property of smooth muscles and tubes that carry fluid through vessels as a result of motor activity in numerous biological systems. The governing equations for continuity and motion have been constructed, and analytic solutions have been performed using the assumptions of a long wave-length and a low Reynolds number. The effect of emerging parameters on the velocity and pressure could be studied and the phenomenon of trapping also discussable.

2 Mathematical Formulation for Asymmetric Flow

Consider the flow of an incompressible "Powell-Eyring fluid" in a two-dimensional asymmetric channel of width (d + d'). The flow is caused by an infinite sinusoidal wave line moving forward and with constant velocity c along the channel's walls . An asymmetric channel is formed by varying wave amplitudes, phase angles, and channel widths.

The walls geometries get modeled as

$$\overline{h_1}(\bar{x},\bar{t}) = d - a_1 \sin[\frac{2\pi}{\lambda}(\bar{x} - c\bar{t})] \text{ upper wall} \quad (1)$$

$$\overline{h_2}(\bar{x},\bar{t}) = -d' - a_2 \sin[\frac{2\pi}{\lambda}(\bar{x} - c\bar{t}) + \emptyset] \text{ lower wall} \quad (2)$$

Where (a_1), (a_2),(d),(d'), (c) , (t) are the wave amplitudes, channel width, wavelength, and wave speed, ($0 \leq \emptyset \leq \pi$) is the phase difference and thus the rectangular coordinate system gets chosen with the ($\overline{X}$ -axis) parallel to the wave propagation direction and the ($\overline{Y}$ - axis) perpendicular to the wave propagation direction. It's worth noting that ($\emptyset$=0) corresponds to a symmetric channel with out of phase waves, whereas($\emptyset$= π) corresponds to a symmetric channel with in phase waves. (a_1), (a_2), (d), (d') and ($\emptyset$) also meet the following criteria . It is noticed that ($\emptyset$=0) corresponds to symmetric channel with waves out of phase and for ($\emptyset$= π) the waves are in phase . Further (a_1), (a_2) , (d), (d′) and ($\emptyset$) satisfy the condition :

$a_1^2 + a_2^2 + 2a_1 a_2 \cos(\emptyset) \leq (d + d')^2$.

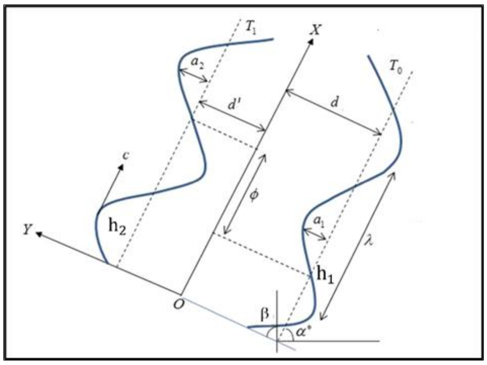

Fig. 10. Cartesian Dimensional Inclined Asymmetric Channels Coordinates.

It's also assumed that there's no longitudinal movement of the walls. This assumption limits wall deformation but does not imply that the channel is stiff for longitudinal motions.

3 Basic equation

The fluid follows the Powell Erring model, and the Cauchy stress tensor,. of it is as follows: [9] .

$$\bar{\tau} = -\mathrm{PI} + \bar{S}, \tag{3}$$

$$\bar{S} = \left[\mu + \frac{1}{\beta\dot{\gamma}} \sinh^{-1}\left(\frac{\dot{\gamma}}{c_1}\right)\right] A_{11}, \tag{4}$$

$$\dot{\gamma} = \sqrt{\frac{1}{2}\mathrm{tras}(A_{11})^2} \tag{5}$$

$$A_{11} = \nabla\bar{V} + (\nabla\bar{V})^T \tag{6}$$

Where $(\bar{S})$expresses the extra tensor's stress, I the identity tensor, $\bar{\nabla}=(\partial\bar{X}, \partial\bar{Y}, 0)$ the gradient vector, (β, c_1) the Powell-Eyring fluid's martial characteristics, $(\bar{P})$ the fluid's pressure, and (μ) the dynamic viscosity.

The terms $\sinh^{-1}$ is approximated as

$$\sinh^{-1}\left(\frac{\dot{\gamma}}{c_1}\right) = \frac{\dot{\gamma}}{c_1} - \frac{\dot{\gamma}^3}{6\,{c_1}^3}, \left|\frac{\dot{\gamma}^5}{{c_1}^5}\right| \ll 1 \tag{7}$$

$$\bar{s}_{\bar{x}\bar{x}} = 2\left(\mu + \frac{1}{\beta\, c_1}\right)\bar{u}_{\bar{x}} - \frac{1}{3\,\beta\, {c_1}^3}\left[2{\bar{u}_{\bar{x}}}^2 + \left(\bar{v}_{\bar{x}} + \bar{u}_{\bar{y}}\right)^2 + 2{\bar{v}^2}_{\bar{y}}\right]\bar{u}_{\bar{x}}, \tag{8}$$

$$\bar{s}_{\bar{x}\bar{y}} = \left(\mu + \frac{1}{\beta\, c_1}\right)(\bar{v}_{\bar{x}} + \bar{u}_{\bar{y}}) - \frac{1}{6\,\beta\, {c_1}^3}\left[2{\bar{u}_{\bar{x}}}^2 + \left(\bar{v}_{\bar{x}} + \bar{u}_{\bar{y}}\right)^2 + 2{\bar{v}^2}_{\bar{y}}\right](\bar{v}_{\bar{x}} + \bar{u}_{\bar{y}}), \tag{9}$$

And $\bar{s}_{\bar{y}\bar{y}} = 2\left(\mu + \frac{1}{\beta\, c_1}\right)\bar{v}_{\bar{y}} - \frac{1}{3\,\beta\, {c_1}^3}\left[2{\bar{u}_{\bar{x}}}^2 + \left(\bar{v}_{\bar{x}} + \bar{u}_{\bar{y}}\right)^2 + 2{\bar{v}^2}_{\bar{y}}\right]\bar{v}_{\bar{y}}.$ (10)

4 The governing equation

With in laboratory frame $(\bar{x}, \bar{y})$, the governing equations inside an inclined channel with inclined magnetic field on Powel-Eyring fluid can be written as the continuous equation:

$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0.$ (11)

The $\bar{x}$ – component of moment equation :

$$\rho\left(\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial}{\partial \bar{x}}\bar{s}_{\bar{x}\bar{x}} + \frac{\partial}{\partial \bar{y}}\bar{s}_{\bar{x}\bar{y}} - \sigma{\beta_0}^2\cos\beta(\bar{u}\cos\beta - \bar{v}\sin\beta) - \frac{\mu}{\bar{k}}\bar{u} - \mu_1 \nabla^4 \bar{u} + \rho g \sin\alpha^* \quad (12)$$

The $\bar{y}$ – component of moment equation :

$$\rho\left(\frac{\partial \bar{v}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\right) = -\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\partial}{\partial \bar{x}}\bar{s}_{\bar{x}\bar{y}} + \frac{\partial}{\partial \bar{y}}\bar{s}_{\bar{y}\bar{y}} - \sigma{\beta_0}^2\sin\beta(\bar{u}\cos\beta - \bar{v}\sin\beta) - \frac{\mu}{\bar{k}}\bar{v} - \mu_1 \nabla^4 \bar{v} - \rho g \cos\alpha^*. \quad (13)$$

Let $\nabla^2 = \left(\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2}\right)$ then $\nabla^4 = (\nabla^2)^2$

where the (ρ), $(\bar{u})$, $(\bar{v})$, $(\bar{y})$, $(\bar{p})$, (μ), $(\bar{k})$, (B_0) are the fluid density ,axial velocity , transverse velocity, transverse coordinate , pressure, viscosity , material constant, permeability parameter, constant magnetic field, is the electrical conductivity.

This flow is unsteady with in laboratory frame $(\bar{x}, \bar{y})$, whereas the motion is steady inside a coordinate system flowing there at wave speed (c) in the wave frame $(\bar{x}, \bar{y})$.

5 Dimensionless parameter

We setup the following non-dimensional quantities to perform the non-dimensional analysis:

$$x = \frac{1}{\lambda}\bar{x},\ y = \frac{1}{d}\bar{y},\ u = \frac{1}{c}\bar{u},\ v = \frac{1}{\delta c}\bar{v},\ p = \frac{d^2}{\lambda \mu c}\bar{p},\ t = \frac{c}{\lambda}\bar{t},\ h_1 = \frac{1}{d}\overline{h_1},\ h_2 = \frac{1}{d}\overline{h_2},\ \delta = \frac{d}{\lambda},\ \emptyset = \frac{b}{d},\ Re = \frac{\rho c d}{\mu},\ Ha = d\sqrt{\frac{\sigma}{\mu}}\beta_0,\ Da = \frac{\bar{k}}{d^2},\ w = \frac{1}{\mu \beta c_1},\ A = \frac{w}{6}\left(\frac{c}{c_1 d}\right)^2,\ \alpha = d\sqrt{\frac{\mu}{\mu_1}},\ Fr = \frac{c^2}{dg},\ s_{xx} = \frac{\lambda}{\mu c}\bar{s}_{\bar{x}\bar{x}},\ s_{xy} = \frac{d}{\mu c}\bar{s}_{\bar{x}\bar{y}},\ s_{yy} = \frac{d}{\mu c}\bar{s}_{\bar{y}\bar{y}},\ \beta_1 = \frac{\beta^*}{d}. \quad (14)$$

where (δ) is the wave number, (Ha) is the Hartman number, (Da) Darcy number, (Re) is the Renold number, (Fr) Froude Number, $(\emptyset)$ is the amplitude ratio, (w) is the dimensionless permeability of the porous medium parameter, (w, A) material fluid parameters, (α) couple stress, (α^*) Inclination angle of the channel to the horizontal axis, (β_1) represent the dimensionless slip parameters.

Then, in view of Eq. (14), Eq. (1),(2),and (8) to (13) take the form :

$$h_1(x,t) = 1 - a\sin X, \quad (15)$$

$$h_2(x,t) = -d^* - b\sin\ [X + \emptyset]. \tag{16}$$

$$s_{xx} = 2(1+w)\frac{\partial u}{\partial x} - 2A\left[2\delta^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\delta^2\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + 2\delta^2\left(\frac{\partial v}{\partial y}\right)^2\right]\frac{\partial u}{\partial x} \tag{17}$$

$$s_{xy} = (1+w)\left(\delta^2\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) - A\left[2\delta^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\delta^2\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + 2\delta^2\left(\frac{\partial v}{\partial y}\right)^2\right]\left(\delta^2\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{18}$$

$$s_{yy} = 2(1+w)\ \delta\frac{\partial v}{\partial y} - 2A\delta\left[2\delta^2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\delta^2\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + 2\delta^2\left(\frac{\partial v}{\partial y}\right)^2\right]\frac{\partial v}{\partial y} \tag{19}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{20}$$

$$\mathrm{Re}\ \delta\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \delta^2\frac{\partial}{\partial x}s_{xx} + \frac{\partial}{\partial y}s_{xy} - Ha^2\cos\ \beta\,(u\cos\beta - \delta v\sin\ \beta) - \frac{1}{Da}u - \frac{1}{\alpha^2}\left(\delta^4\ \frac{\partial^4}{\partial x^4} + 2\delta^2\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}\right)u + \frac{\mathrm{Re}}{\mathrm{Fr}}\ \sin\alpha^* \tag{21}$$

$$\mathrm{Re}\ \delta^3\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \delta^2\frac{\partial}{\partial x}s_{xy} + \delta\frac{\partial}{\partial y}s_{yy} + Ha^2\sin\ \beta\,(\delta u\cos\ \beta - \delta^2\ v\sin\ \beta) - \delta^2\frac{1}{Da}v - \frac{1}{\alpha^2}\delta^2\left(\delta^4\ \frac{\partial^4}{\partial x^4} + 2\delta^2\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}\right) - \ \delta\frac{\mathrm{Re}}{\mathrm{Fr}}\cos\alpha^* \tag{22}$$

The relations connect the stream function (ψ) to the velocity components.

$$u = \frac{\partial\psi}{\partial y},\ v = -\frac{\partial\psi}{\partial x}, \tag{23}$$

Substituted Eqs.(23) in Eqs. (17) to Eqs. (22) respectively,

$$s_{xx} = 2(1+w)\frac{\partial^2\psi}{\partial x\,\partial y} - 2A\left[2\delta^2\left(\frac{\partial^2\psi}{\partial x\,\partial y}\right)^2 + \left(-\delta^2\ \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)^2 + 2\delta^2\left(-\ \frac{\partial^2\psi}{\partial x\,\partial y}\right)^2\right]\frac{\partial^2\psi}{\partial x\,\partial y} \tag{24}$$

$$s_{xy} = (1+w)\left(-\delta^2\ \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - A\left[2\delta^2\left(\frac{\partial^2\psi}{\partial x\,\partial y}\right)^2 + \left(-\delta^2\ \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)^2 + 2\delta^2\left(-\ \frac{\partial^2\psi}{\partial x\,\partial y}\right)^2\right]\left(-\delta^2\ \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \tag{25}$$

$$s_{yy} = -2(1+w)\delta\frac{\partial^2\psi}{\partial x\,\partial y} - 2A\delta\left[2\delta^2\left(\frac{\partial^2\psi}{\partial x\,\partial y}\right)^2 + \left(-\delta^2\ \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right)^2 + 2\delta^2\left(-\ \frac{\partial^2\psi}{\partial x\,\partial y}\right)^2\right]\frac{\partial^2\psi}{\partial x\,\partial y} \tag{26}$$

$$\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial^2\psi}{\partial x\,\partial y} = 0 \tag{27}$$

$$\mathrm{Re}\,\delta\left(\frac{\partial^2\psi}{\partial t\,\partial y}+\frac{\partial^3\psi}{\partial x\,\partial y^2}-\frac{\partial^3\psi}{\partial x\,\partial y^2}\right)=-\frac{\partial p}{\partial x}+\delta^2\frac{\partial}{\partial x}s_{xx}+\frac{\partial}{\partial y}s_{xy}-\mathrm{Ha}^2\cos\beta\left(\frac{\partial\psi}{\partial y}\cos\beta\ +\right.$$
$$\left.\delta\frac{\partial\psi}{\partial x}\sin\beta\right)-\frac{1}{\mathrm{Da}}\frac{\partial\psi}{\partial y}-\frac{1}{\alpha^2}\left(\delta^4\frac{\partial^4}{\partial x^4}+2\delta^2\frac{\partial^4}{\partial x^2\,\partial y^2}+\frac{\partial^4}{\partial y^4}\right)\frac{\partial\psi}{\partial y}+\frac{\mathrm{Re}}{\mathrm{Fr}}\sin\alpha^* \tag{28}$$

$$\mathrm{Re}\,\delta^3\left(-\frac{\partial^2\psi}{\partial t\,\partial y}+\frac{\partial^3\psi}{\partial x^2\,\partial y}-\frac{\partial^3\psi}{\partial x^2\,\partial y}\right)=-\frac{\partial p}{\partial y}+\delta^2\frac{\partial}{\partial x}s_{xy}+\delta\frac{\partial}{\partial y}s_{yy}+$$
$$\mathrm{Ha}^2\sin\beta\left(\delta\frac{\partial\psi}{\partial y}\cos\beta+\delta^2\frac{\partial\psi}{\partial x}\sin\beta\right)+\delta^2\frac{1}{\mathrm{Da}}\frac{\partial\psi}{\partial x}+\frac{1}{\alpha^2}\delta^2\left(\delta^4\frac{\partial^4}{\partial x^4}+2\delta^2\frac{\partial^4}{\partial x^2\,\partial y^2}+\right.$$
$$\left.\frac{\partial^4}{\partial y^4}\right)\frac{\partial\psi}{\partial x}-\delta\frac{\mathrm{Re}}{\mathrm{Fr}}\cos\alpha^* \tag{29}$$

The wave frame's dimensionless boundary conditions are [10]:

$\psi=\frac{F}{2}$, at y= h_1 , $\psi=-\frac{F}{2}$, at y= h_2,

$\frac{\partial\psi}{\partial y}+\beta_1\frac{\partial^2\psi}{\partial y^2}=-1$ at y= h_1, $\frac{\partial\psi}{\partial y}-\beta_1\frac{\partial^2\psi}{\partial y^2}=-1$ at y= h_2,

$$\frac{\partial^3\psi}{\partial y^3}=0 \text{ at } y=h_1,\quad \frac{\partial^3\psi}{\partial y^3}=0 \text{ at } y=h_2. \tag{30}$$

In the wave frame, (F) is the dimensionless temporal mean flow rate. Through the expression, it is related to the dimensionless temporal mean flow rate (Q) in the laboratory frame. [58]

$$Q=F+1+d^* . \tag{31}$$

$h_1(x)$ and $h_2(x)$ have dimensionless forms:

$$h_1(x)=1+a\sin(X),\quad h_2(x)=-d^*-b\sin(X+\emptyset) \tag{32}$$

where (a) , (b) ,($\emptyset$) and (d^*) satisfy [10]:
$a^2+b^2+2\,ab\cos(\emptyset)\le(1+d^*)^2$.

6 Effect of couple - stress

A relationship here between couple stress parameter (α) and the material fluid parameters (A) would be discovered in this section.

This relationship will aid us in simplifying the problem's solution strategy. Because, as mentioned in the previous chapter, finding the zero and first-order solutions is required seeing the effect of any and all parameters that present in the problem. However , using the relationship between the couple stress parameter and the material fluid parameters we need to find the zero order only .

From dimensionless the material fluid parameters :

Let $A=\frac{w}{6}\left(\frac{c}{c_1\ d}\right)^2$

then

$$d = \sqrt{\frac{w}{6A}}\left(\frac{c}{c_1}\right), \tag{33}$$

$$\text{since} \quad \alpha = d\sqrt{\frac{\mu}{\mu_1}} \tag{34}$$

substitute Eq.(34) into Eq.(34), we get $\alpha = \sqrt{\frac{w\mu}{6A\mu_1}}\left(\frac{c}{c_1}\right)$

$$\alpha^2 = \frac{w\mu}{6A\mu_1}\left(\frac{c}{c_1}\right)^2 \quad \text{and} \quad \frac{1}{\alpha^2} = \frac{6A\mu_1}{w\mu}\left(\frac{c_1}{c}\right)^2 \tag{35}$$

7 Solution of the problem

Substitute the terms (31) in to Eqs. (24) to Eqs. (29), together with the boundary conditions Eqs. (30) Since $\delta \leq 1$, and using the approximation of a long wavelength and a low Reynolds number. For the appearance of the couple stress parameter in the equation, the solution is limited to the zero order by giving all the parameters required to solve the problem and find the results, we get the motion equation in the terms of stream function which is

$$\psi_{yyyy} - \xi\,\psi_{yy} - \frac{1}{\alpha^2}\psi_{yyyyyy} = 0. \tag{36}$$

$$\xi = \frac{Ha^2\cos^2\beta + \frac{1}{Da}}{w+1} \tag{37}$$

$$\eta = \frac{1}{1+w} \tag{38}$$

The solution of the momentum equation is straight forward and can be written as

$$\psi = \sqrt{2}\left(\frac{\sqrt{2}e^{\frac{y\sqrt{\frac{\alpha\left(\alpha-\sqrt{\alpha^2-4\zeta\eta}\right)}{\eta}}}{\sqrt{2}}}\eta c1}{\alpha\left(\alpha-\sqrt{\alpha^2-4\zeta\eta}\right)} + \frac{\sqrt{2}e^{-\frac{y\sqrt{\frac{\alpha\left(\alpha-\sqrt{\alpha^2-4\zeta\eta}\right)}{\eta}}}{\sqrt{2}}}\eta c2}{\alpha\left(\alpha-\sqrt{\alpha^2-4\zeta\eta}\right)} + \frac{\sqrt{2}e^{\frac{y\sqrt{\frac{\alpha\left(\alpha+\sqrt{\alpha^2-4\zeta\eta}\right)}{\eta}}}{\sqrt{2}}}\eta c3}{\alpha\left(\alpha+\sqrt{\alpha^2-4\zeta\eta}\right)} + \frac{\sqrt{2}e^{-\frac{y\sqrt{\frac{\alpha\left(\alpha+\sqrt{\alpha^2-4\zeta\eta}\right)}{\eta}}}{\sqrt{2}}}\eta c44}{\alpha\left(\alpha+\sqrt{\alpha^2-4\zeta\eta}\right)}\right) + c5 + yc6 \tag{39}$$

From Eq. (25) in Eq. (28) we get :

$$\frac{\partial p}{\partial x} = (w+1)\psi_{yyy} - (w+1)\xi\,\psi_y - \frac{1}{\alpha^2}\psi_{yyyyy} + \frac{Re}{Fr}\sin\propto^*. \tag{40}$$

$$-\frac{\partial p}{\partial y} = 0 \tag{41}$$

The pressure rise per wave length (Δp) is defined as

$$\Delta p = \int_0^1 \frac{\partial p}{\partial x} dx \ . \qquad (42)$$

In the fixed frame, this axial velocity component is given as

$$u(x., y., t.)=1+ \psi_y \qquad (43)$$

8 Results and Discussion

To study the effect of physical parameters such as Effect of Hartman number (Ha), Darcy number (Da), Renold number (Re), Froude number (**Fr**) ,couple stress (**α**), Inclination angle of the channel to the horizontal axis ($\boldsymbol{\alpha}^*$), inclination of magnetic field (**β**), represent the dimensionless slip parameters ($\boldsymbol{\beta_1}$), material fluid parameter (w) and amplitude ratio (∅). we have plotted the axial velocity (u), and stream function (**ψ**) in figs. 2.-15. are illustrated using the software MATHEMATICA .

8.1 Velocity distribution

For varying values of (u), difference in axial velocity throughout the channel . The effect different values of (Ha), (Da), (β), (β_1),(w),(α) and (∅) on axial velocity (u) are explained in Figs. 2.- 8. The behavior of velocity distribution is parabolic as seen in figures. Figs. 2.,5. shows that the axial velocity with increasing (Ha) and (β_1) increases in the central region and the boundary of the channel wall. Fig.3. displayed the influence of (Da) on the axial velocity, it is noticed that at the walls of the channel, the axial velocity decreases with an increase of (Da), and decreases at the center of the channel. Fig. 4. noted that the axial velocity do not change at increasing in (β). Figs 6.,7. the axial velocity increasing with increasing (α) and (w) increasing in the central region and not change in the boundary of the channel wall. From fig.8. At increasing in (∅), the axial velocity falls in the middle region and the channel's boundary right, while increasing in the channel's boundary left.

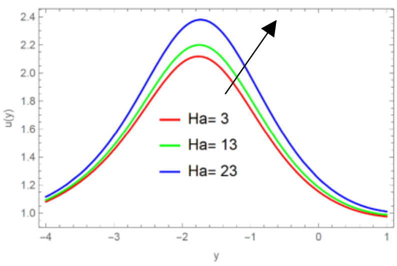

Fig. 10. Variation of velocity for different values of Ha when Da=3, β=0.5, β_1=4, α= 0.4, w= 3, $\emptyset$= 0.5 ,a= 0.2 , b= 0.2 , d^*= 0.5 .

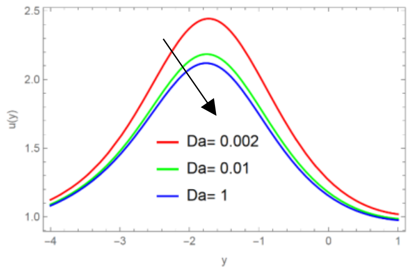

Fig. 9. Variation of velocity for different values of Da when Ha=3, β=0.5, β_1=4 , α=0.4, w= 3, $\emptyset$= 0.5 ,a= 0.2 , b= 0.2, d^*= 0.5 .

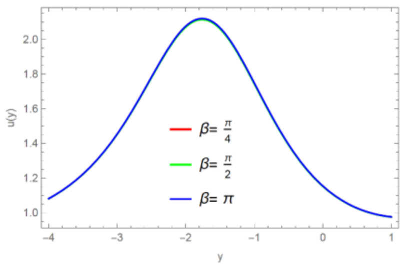

Fig. 11. Variation of velocity for different values of β when Ha=3 , Da=3, β_1=4, α= 0.4, w= 3 , $\emptyset$= 0.5 ,a= 0.2 , b= 0.2 , d^*= 0.5

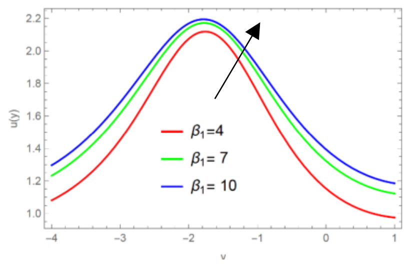

Fig. 5. Variation of velocity for different values of β_1 when Ha=3,Da=3, β=0.5, α= 0.4, w= 3 , $\emptyset$= 0.5 ,a= 0.2 , b= 0.2 , d^*= 0.5 .

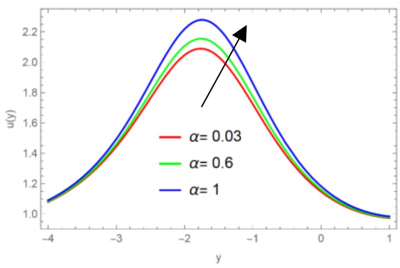

Fig. 13. Variation of velocity for different values of α when Ha=3 ,Da=3, β=0.5, β_1=4 ,w= 3 , $\emptyset$= 0.5 ,a= 0.2 , b= 0.2 , d^*= 0.5 .

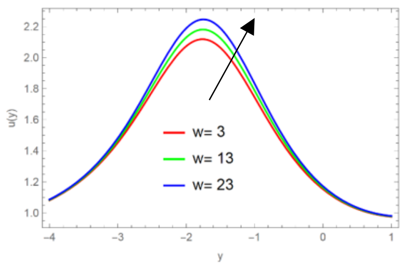

Fig. 14. Variation of velocity for different values of w when Ha=3 , Da=3, β=0.5, β_1=4 , α= 0.4, $\emptyset$= 0.5 ,a= 0.2 ,b= 0.2 , d^*= 0.5 .

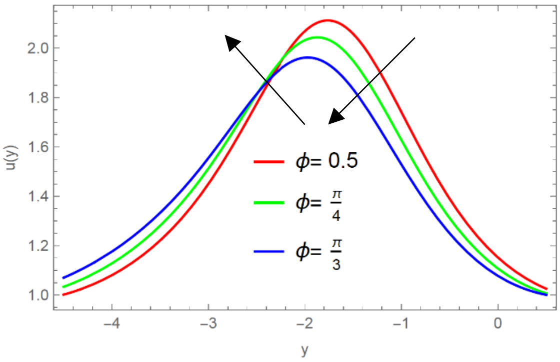

Figure 8. Variation of velocity for different values of $\emptyset$ when Ha=3, Da=3, β=0.5, β_1=4, α= 0.4, w= 2,a= 0.2 , b= 0.2 , d^*= 0.5 .

8.2 Trapping phenomenon

Closed stream lines trap the amount of fluid known as bolus inside the channel tube near the walls in peristaltic flows, and this trapped bolus pushes forward in the direction of wave propagation. In figs 9. – 15. the stream lines are plotted at various values of Ha, Da, $\boldsymbol{\beta}$, $\boldsymbol{\beta_1}$, $\boldsymbol{\alpha}$,w and $\emptyset$. Figs 9. , 14. the exhibits that the trapping exist for both upper and lower walls , we observe that size of trapping bolus decreases with increases (Ha) and (w) . Figs 10. and 11. the exhibits that the trapping exist for both upper and lower walls , we observe that size of trapping bolus no change with increases (Da) and ($\boldsymbol{\beta}$) . Figs 12. , 13. show that trapping exists for both the upper and lower walls, and that the size of the trapping bolus lowers and expands as ($\boldsymbol{\beta_1}$) and ($\boldsymbol{\alpha}$) increase. Fig 15. the exhibits that the trapping exist for both upper and lower walls, we observe that size of trapping bolus increases with increases ($\emptyset$) and open channel with ($\emptyset = \boldsymbol{\pi}$) .

(a) **(b)** **(c)**

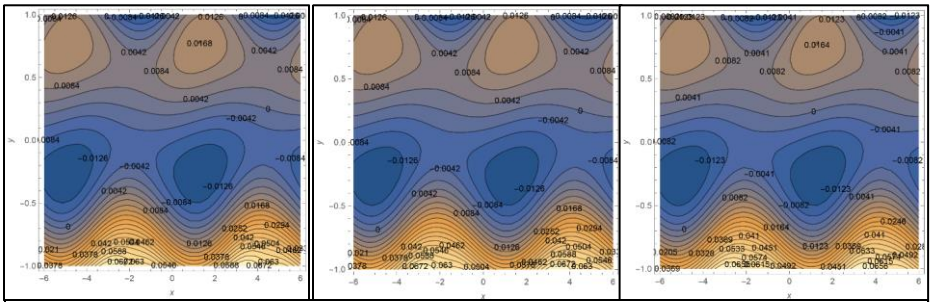

Fig. 15. Stream function in the wave frame of **Ha** such that in (a)Ha=3,(b) Ha=6,(c)Ha=9, in Da =2, β=0.5, β_1 =4, $\alpha = 0.4$, w=2, $\emptyset = 0.5$, $a = 0.2$, $b = 0.2$, $d^* = 0.5$.

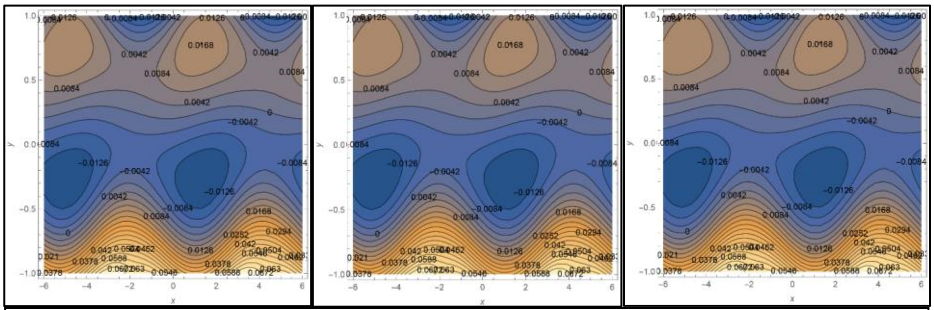

Fig. 10. Stream function in the wave frame of **Da** such that in (a) Da=2, (b) Da=4, (c) Da=6, in Ha=3, β=0.5, β_1 =4, $\alpha = 0.4$, w=2, $\emptyset = 0.5$, $a = 0.2$, $b = 0.2$, $d^* = 0.5$.

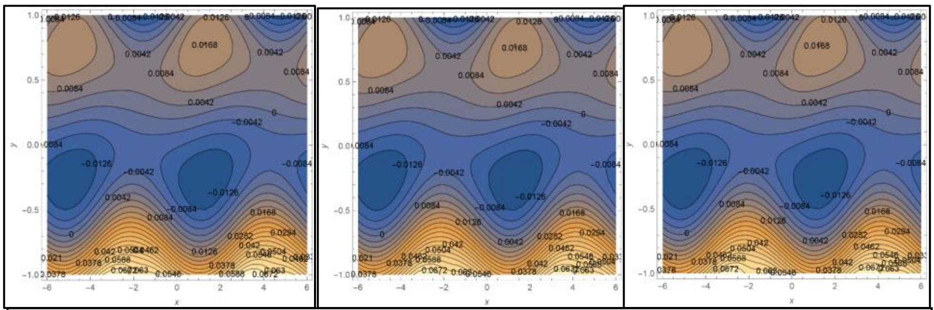

Fig. 11. stream function in the wave frame of $\boldsymbol{\beta}$ such that in (a) β =0.5, (b) $\beta = \frac{\pi}{3}$, (c) $\beta = \frac{\pi}{2}$, in Ha =3, $Da = 2$, $\beta_1 = 4$, $\alpha = 0.4$, $w = 2$, $\emptyset = 0.5$, $a = 0.2$, $b = 0.2$, $d^* = 0.5$.

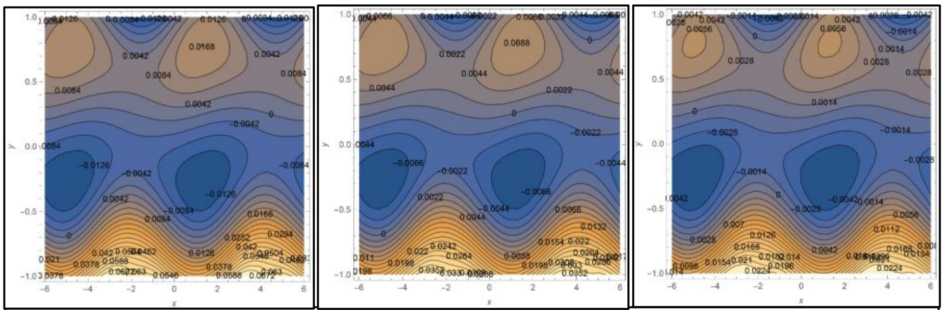

Fig. 12. stream function in the wave frame of $\boldsymbol{\beta_1}$ such that in (a) β_1 =4, (b) β_1 =8 , (c) β_1=12 , in Ha =3, $Da = 2$, $\beta = 0.5$, $\alpha = 0.4$, $w = 2$, $\emptyset = 0.5$, $a = 0.2$, $b = 0.2$, $d^* = 0.5$.

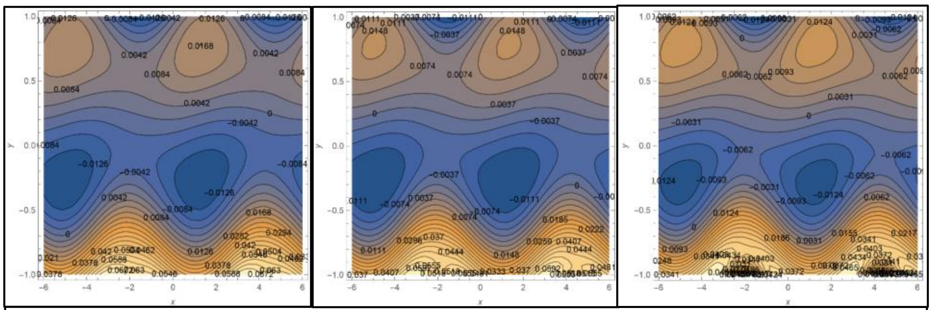

Fig. 13 stream function in the wave frame of **α** such that in (a) α =0.4 , (b) α =1.6 , (c) α =2.8 , in Ha =3, Da =2, β=0.5, β_1 =4, w=2, $\emptyset = 0.5, a = 0.2, b = 0.2, d^* = 0.5$.

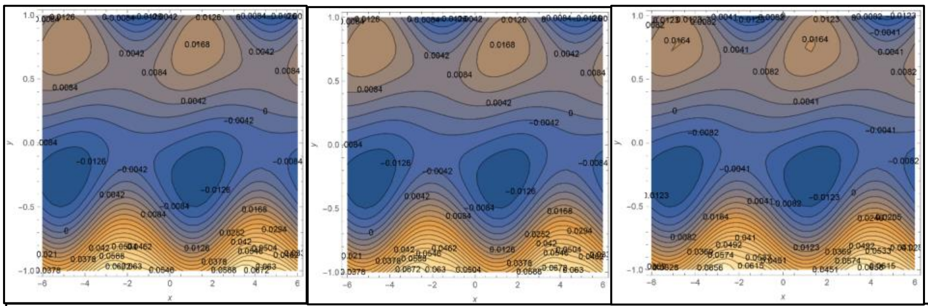

Fig. 14. stream function in the wave frame of **w** such that in (a)w= 2, (b)w =6, (c)w=10 in Ha =3 , Da =2, β=0.5, β_1 =4, $\alpha = 0.4, \emptyset = 0.5, a = 0.2, b = 0.2, d^* = 0.5$.

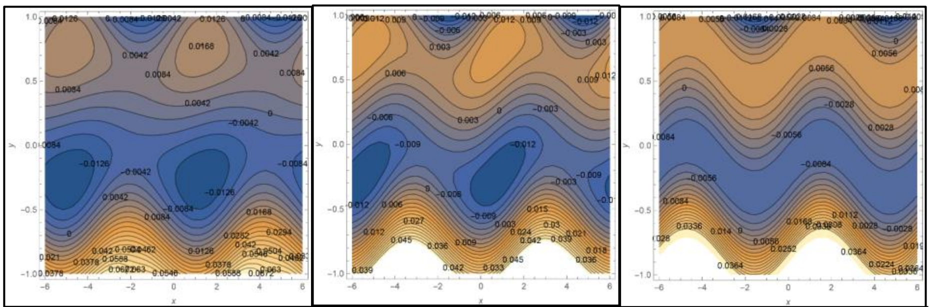

Fig. 15. stream function in the wave frame of **∅** such that in (a)∅=0.5,(b)∅= $\frac{\pi}{2}$, (c)∅=π in Ha =3 , Da =2, β=0.5, β_1 =4, $\alpha = 0.4$, w=2, $a = 0.2, b = 0.2, d^* = 0.5$.

9 Conclusions

In light of this studies, some of the more intriguing findings have been described, with a focus on the study Effect of Couple Stress on Peristaltic Transport of Powell-

Eyring Fluid Peristaltic flow in Inclined Asymmetric Channel with Porous Medium The results are discussed through graphs , as follows :

- By increasing (Ha) and (β_1) increases in the central region and the boundary of the channel wall but the opposite occur for increasing (Da).
- The axial velocity no change near the wall while it increases at the center of the channel by increasing (α) and (w) . Furthermore increasing (β) has not effected on the axial velocity.
- At increasing in ($\emptyset$) , the axial velocity falls in the middle region and the channel's boundary right, while increasing in the channel's boundary left.
- The size of trapped bolus decreases with increasing (Ha) and (w) , we observe that size of trapping bolus no change with increases (Da) and (β) .
- Demonstrate that trapping exists for both the upper and lower walls, and that the size of the trapping bolus decreases and increases as (β_1) and (α) rise .
- The exhibits show that the trapping is present on both the upper and lower sides, we observe that size of trapping bolus increases with increases ($\emptyset$) and open channel with ($\emptyset = \pi$) .

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Article submitted 16 June 2022. Published as resubmitted by the authors 1 August 2022.