On Hyperbolic Differential Equation with Periodic Control Initial Condition

Zainab Abdel Ameer
Al-Mustansiriya University, College of Education, Math Dep. Iraq.
zynbaldial@gmail.com

Sameer Qasim Hasan
Al-Mustansiriya University, College of Education, Math Dep. Iraq.

Abstract. The aim of this paper to study the existence solution of some types of hyperbolic differential equation with periodicity of some controls function as nonlocal initial condition for the equation and the technical that are used with analytic depended on some interest iniquities, make advantage steps for proving with fixed point theorems to grantees of the solution.

Keyword. Periodic hyperbolic equation, control initial functions, periodic solution.

1. Introduction

In this section we assume the hyperbolic deferential with periodic control initial conditions, as follow:

\[ u_{tt} - \mathrm{div}(\sigma(|\nabla u|^2) \nabla u^m) - (\nabla u)^p = |u|^{m-2} f(x, t, u) \log|u|, \quad x \in \Omega, t > 0, \quad (1.1) \]
\[ u(x, t) = 0, x \in \partial \Omega, \quad t \in [0, 1], \quad (1.2) \]
\[ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega, \quad (1.3) \]

where \( u_0, u_1 \in L^2(\Omega) \), are periodic control functions, \( \Omega \) is in a bounded domain \( \mathbb{R}^n \) with \( \partial \Omega \) as smooth bounded \( Q_w = R \times (0, T] \), for \( [0, 1] \). The term \( u^{m-2} \log|u| \) is a logarithmic nonlinearity which can be applied to many branches [1], [2], [3]. Where \( p > 0 \) and \( m \geq 1 \).

Assume the following conditions:

a. \( f(x, t, u) \) is Holder continuous in \( \bar{\Omega} \times R \times R \), periodic in \( t \) with a period \( T \) and satisfied \( f(x, t, u) \leq b_0|u|^\alpha \) with constant \( b_0 \geq 0 \) and \( 0 \leq \alpha \leq 1 \).

b. \( \sigma(|\nabla u|^2) = |\nabla u|^m \).

c. \( \mathrm{div}(\sigma(|\nabla u|^2) \nabla u^m) \) is the definition of rather slow.

There are many articles researched on some types of hyperbolic differential equations with some properties and important results such as in [4] the multipoint boundary value problems with nonlocal initial conditions for hyperbolic defferential problems with different approaches. Impulsive System with Periodic Problem for a hyperbolic equation in [5] has been studied extensively. The loaded third order equations as hyperbolic boundary value problems also with mixed types were presented in [6] and in [7] the optimal boundary control problems of the turnpike phenomenon corresponding to hyperbolic systems have been studied and computed. Finely in [8, 9, 10] Strict Lyapunov Function used to studied a boundary control of hyperbolic systems.

In this paper, the existence and unique solution of the hyperbolic problem with periodic functions as initial conditions (1.1) - (1.3), and shown that the solution is uniformly bounded, we establish the existence under some conditions are sufficiently conditions to satisfy Leray –Schauder fixed point theorem.
2. Main results

In this section the results of explain existence of solution for the hyperbolic deferential of problem with initial periodic control functions now, we need the following definition.

**Definition (2.1):** A function \( v \in L^p (0, w; w_0^{1,p}(\Omega)) \cap C_w (Q_w) \) is called to be a weak solution of the problem (1.1) - (1.3) if \( \frac{\partial^2 v}{\partial t^2} \in L^2(Q_w) \) and \( v \) satisfies

\[
\int_{Q_T} \left( \frac{\partial^2 v}{\partial t^2} \varphi + m v^{1-\frac{1}{m}} \sigma(|\nabla u|^2) \nabla v \nabla \varphi \right) + (m - 1) v^{\frac{1}{m}} \sigma(|\nabla u|^2) \varphi - |v|^{m-2} f(\alpha, t, v) \log |v| \varphi dx \ dt = 0,
\]

for any \( \varphi \in C^2(\overline{Q}_w) \) with \( \varphi(x, 0) = u_0(t), \varphi(x, 0) = u_1(t) \)

And \( \varphi \log \Omega \times (0, t) = 0 \)

**Theorem (2.1):** Let \( u \) be solution of \( u_{tt} - \div (v (|\nabla u|^2) \nabla u) - (\nabla u)^p = |u|^{m-2} f(x, t, u) \log |u| \)

\( u(x, t) = 0, x \in \partial \Omega, \ t \in [0, 1], \ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \in \Omega, \) with \( \sigma \in [0, 1] \), then there exists positive constant \( R \) independent of \( \sigma \) such that \( ||u(t)||_2^2 < R. \)

Proof:

\[
\frac{1}{(m+2)(m+3)} \frac{\partial^2 u}{\partial t^2} m + 1 = \frac{1}{m+2} \left[ (m + 2)u^{m+1} \left( \frac{\partial u}{\partial t} \right)^2 + u^{m+2} \frac{\partial^2 u}{\partial t^2} \right] = u^{m+1} \nabla u + \frac{u^{m+2}}{m+2} \frac{\partial^2 u}{\partial t^2}
\]

Thus

\[
\frac{1}{m+3} \frac{\partial^2 u}{\partial t^2} m + 1 = \frac{1}{m+2} \left( (m + 2)u^{m+1} \nabla u + \frac{\partial^2 u}{\partial t^2} \right)^2
\]

\[
\frac{1}{m+3} \frac{\partial^2 u}{\partial t^2} m + 2 = (m + 2)u^{m+1} \nabla u + \frac{\partial^2 u}{\partial t^2}
\]

Multiply Eq. (1.1) by \( u^{m+2} \), we get

\[
\int_{\Omega} \frac{1}{m+2} \frac{\partial^2 u}{\partial t^2} u^{m+2} - (m + 2)u^{m+1} \nabla u + \frac{\partial^2 u}{\partial t^2}
\]

Multiply Eq. (1.1) by \( |u(t)|^p \), \( p > 0 \), and integrating, we get

\[
\int_{\Omega} \frac{1}{m+2} \frac{\partial^2 u}{\partial t^2} u^{m+2} |u(t)|^p - \int_{\Omega} (m + 2)u^{m+2} \nabla u |u(t)|^p + \int_{\Omega} \sigma (|\nabla u|^2) |u(t)|^p \nabla u - |u(t)|^p (\nabla u)^p + \int_{\Omega} \sigma (|\nabla u|^2) |u(t)|^p \nabla u - |u(t)|^p (\nabla u)^p + \int_{\Omega} \sigma (|\nabla u|^2) |u(t)|^p \nabla u - |u(t)|^p (\nabla u)^p u^{m+3}
\]

\[
= \int_{\Omega} |u|^{m-2} f(x, t, u) |u(t)|^p \nabla u - |u(t)|^p (\nabla u)^p u^{m+2} \log |u|
\]

\[
= \frac{1}{m+3} \frac{\partial^2 u}{\partial t^2} \int_{\Omega} |u|^{m+3} |u(t)|^p - \int_{\Omega} (m + 2) \frac{1}{m+3} \left( \nabla \left( u^{m+3} \ nabla u \right) \right)^p + \int_{\Omega} (p + 1) \sigma (|\nabla u|^2) m u^{m-1} |\nabla u|^2 - \left( u^p \ nabla u \right)^p
\]

221
\[
\begin{align*}
\frac{1}{m+1} \frac{\partial^2}{\partial t^2} \left| u \right|^{m+2} - \left( \frac{m+2}{m+3} \right)^p - \frac{p(m+2)}{m+3+p} \frac{\nabla \left( \frac{u}{m+3+p} \right)}{m+3+p}
\end{align*}
\]

\[
+ (p+1) \int_\Omega \left\{ m \sigma \left( |\nabla u|^2 \right) \left| \frac{m-1}{2} \nabla u \right|^2 - \left| \frac{m+3+p}{u} \right| \right\}
\]

\[
= \int_\Omega \left| u \right|^{p+1} f(x,t,u) log|u|
\]

\[
- \frac{1}{m+1} \frac{\partial^2}{\partial t^2} \left| u \right|^{m+2} - \left( \frac{m+2}{m+3} \right)^p - \frac{p(m+2)}{m+3} \frac{\nabla \left( \frac{u}{m+3} \right)}{m+3}
\]

\[
+ (p+1) \frac{2}{2m+1} \left| \nabla u \right|^{2m+1} \left( \frac{2m}{2m+1} \left| \nabla u \right|^{2m+1} - \left| \frac{p}{m+3+2p} \nabla \frac{m+3+2p}{u} \right|^p \right)
\]

\[
\leq \left| u \right|^{p+1} b_0 \left| u \right|^a log|u|
\]

Therefore,

\[
\frac{1}{m+1} \frac{\partial^2}{\partial t^2} \left| u \right|^{m+2} - \left( \frac{m+2}{m+3} \right)^p - \frac{p(m+2)}{m+3} \frac{\nabla \left( \frac{u}{m+3} \right)}{m+3}
\]

\[
+ (p+1) \left( \frac{2m}{2m+1} \left| \nabla u \right|^{2m+1} - \left| \frac{p}{m+3+2p} \nabla \frac{m+3+2p}{u} \right|^p \right)
\]

\[
\leq \left| u \right|^{p+1} b_0 \left| u \right|^a \log|u|
\]
\[
\left( \frac{2m}{2m+1} \right) \left\| \nabla u^{2m+1} \right\|_2^2 - \left\| \frac{p}{m+3+2p} \nabla u^{m+3+2p} \right\|_p^p \geq b_0 \left\| \frac{2m+6+p+\alpha}{p+1} \right\|_p^p \log |u|.
\]

Suppose that
\[
\left\| \nabla u^{2m+1} \right\|_2^2 < \frac{p(m+2)}{m+3+p} \left\| \nabla \left( \frac{u^{m+3+p}}{p} \right) \right\|_p^p + \frac{(p+1)p}{m+3+2p} \left\| \nabla u^{m+3+p} \right\|_p^p,
\]
and set \( m = p \), we get that,
\[
\frac{1}{p+1} \frac{\partial^2}{\partial t^2} \left\| u^{2p+1} \right\|_p^p + \frac{(p+1)}{2p+1} \left\| \nabla u^{2p+1} \right\|_2^2 \leq b_0 \left\| \frac{2p+6+p+\alpha}{p+1} \right\|_p^p \log |u|
\]
\[
\frac{1}{(p+1)} \frac{\partial^2}{\partial t^2} \left\| u^{2p+1} \right\|_p^p + \frac{2(p+1)p}{2p+1} \left\| \nabla u^{2p+1} \right\|_2^2 \leq \left\| \nabla u^{2p+1} \right\|_2^2
\]
\[
\leq b_0 \left\| \frac{3p+6+\alpha}{p+1} \right\|_p^p \log |u|
\]

Thus,
\[
\frac{1}{(p+1)} \frac{\partial^2}{\partial t^2} \left\| u^{2p+1} \right\|_p^p + \frac{(2p+1)p}{2p+1} \left\| \nabla u^{2p+1} \right\|_2^2 \leq b_0 \left\| \frac{3p+6+\alpha}{p+1} \right\|_p^p \log |u| \tag{1.6}
\]

Let \( u_k = \frac{2p+1}{u_0} \) and \( 0 < \alpha \leq 1 \), we have
\[
\frac{1}{(p+1)} \frac{\partial^2}{\partial t^2} \left\| u^{2p+1} \right\|_p^p + \frac{(2p+1)p}{2p+1} \left\| \nabla u^{2p+1} \right\|_2^2 \leq b_0 \left\| \frac{3p+6+\alpha}{p+1} \right\|_p^p \log |u_k| \tag{1.7}
\]

By the Gagliardo-Nirenbergy inequality, we have that
\[
\left\| u_k(t) \right\|_2^2 \leq C \left\| \nabla u_k(t) \right\|_2^2 \left\| u_k(t) \right\|_2^{-\theta}, \text{ where } \theta = \frac{N}{N+2} \in (0,1)
\]

Noting that \( \left\| u_k(t) \right\|_1 = \left\| u_{k-1}(t) \right\|_2^2 \)
\[
\frac{\partial^2}{\partial t^2} \left\| u_k(t) \right\|_2^2 \leq \left( 1 - \frac{2(pk+1)p}{2p+1} \right) \left( p_k + 1 \right) \left\| \nabla u_k(t) \right\|_2^2 + b_0 (b_k + 1) \left\| u_k \right\|_2^{\frac{2(3p_k+7)}{p_k+1}} \log |u_k| \tag{1.9}
\]

Set \( \lambda_k = \max \{ 1, \sup \epsilon \left\| u_k(t) \right\|_2^2 \} \), then
\[
\frac{\partial^2}{\partial t^2} \left\| u_k(t) \right\|_2^2 \leq \left\| u_k(t) \right\|_2^2
\]

223
\[
\left\{ \left(1 - \frac{2(p_k+1)p_k}{2p_k+1}\right)(p_k+1)\|u_k(t)\|_2^2 - \frac{2}{p_k+1}\lambda_k^{-\theta} + b_0(b_k + 1) \log |u_k| \right\} \tag{1.10}
\]

\[
\frac{\partial^2}{\partial t^2} \|u_k(t)\|_2^2 \leq \|u_k(t)\|_2^2 \frac{2(3p_k+7)}{p_k+1}
\]

\[
\left\{ \left(1 - \frac{2(p_k+1)p_k}{2p_k+1}\right)(p_k+1)\|u_k(t)\|_2^2 - \frac{2}{p_k+1}\lambda_k^{-\theta} + b_0(b_k + 1) \log |u| \right\}
\]

\[
p_{k+1} \frac{\partial^2}{\partial t^2} \|u_k(t)\|_2^2 \leq \left(1 - \frac{2(p_k+1)p_k}{2p_k+1}\right)(p_k+1) \|u_k(t)\|_2^2 \frac{2}{p_k+1}\lambda_k^{-\theta} + b_0(b_k + 1) \log |u_k| \tag{1.11}
\]

Since \(u_1(t), u_2(t)\) are periodic control functions.

\[
\|u_k(t)\|_2 \leq \left\{ \mathcal{C} \left[ b_0(b_k + 1) \log |u_k| + \left(1 - \frac{2(p_k+1)p_k}{2p_k+1}\right)(p_k+1) \right] \right\} \frac{1}{\lambda_k^{-\theta}}
\]

Where \(\mathcal{C}_k = \frac{2}{\theta} - \frac{2(p_k+1)}{p_k+2} = \frac{2\ell_k}{p_k^2+2}
\]

Since \(\lambda_k^{-\theta} \geq 1, k = 1, 2\), we get

\[
\|u_k(t)\|_2 \leq \left\{ \left(1 - \frac{2(p_k+1)p_k}{2p_k+1}\right)(p_k+1) \lambda_k^{b_k+1} \right\} \frac{1}{\lambda_k^{-\theta}}
\]

\[
\|u_k(t)\|_2 \leq \left\{ \left(1 - \frac{2(p_k+1)p_k}{2p_k+1}\right)(p_k+1) \right\} \frac{1}{2\lambda_k^{b_k+1}}
\]

Where \(a'\) is a positive constant independent of \(k\), and \(A = 2a'\)

\[
\ln \|u(t)\|_2 \leq \ln \ell_k \leq \ln \left(1 - \frac{2(p_k+1)p_k}{2p_k+1}\right)(p_k+1) + k \ln A + 2 \ln \ell_k - 1
\]

Where \(A = 2a' > 1\), then

\[
\sum_{i=1}^{k-2} 2^i + 2^{k-1} \ln 1 + \ln A \left( \sum_{j=0}^{k-2} (k-j)2^j \right)
\]

\[
\leq (2^{k-1} - 1) \ln \left(1 - \frac{2(p_k+1)p_k}{2p_k+1}\right)(p_k+1) + 2^{k-1} \ln \lambda_1 + f(k) \ln A , \text{ with}
\]

\[
f(k) = 2^{k+1} - 2^{k-1} - k - 2
\]

That is

\[
\|u(t)\|_{p_{k+2}} \leq \left\{ \left(1 - \frac{2(p_k+1)p_k}{2p_k+1}\right)(p_k+1) \lambda_k^{2k-1} \right\} \frac{1}{\lambda_k^{-\theta}}
\]

Letting \(k \rightarrow \infty\), we get

\[
\|u(t)\|_\infty \leq \left(1 - \frac{2(p_k+1)p_k}{2p_k+1}\right)(p_k+1) \lambda_k^2 \leq \left(1 - \frac{2(p_k+1)p_k}{2p_k+1}\right)(p_k + 1)(\text{max}\{1, \sup_t \|u(t)\|_2\})^2
\]

From (1.7)

\[
\frac{\partial}{\partial t^2} \|u(t)\|_2^2 + \left(\frac{2(p_k+1)p_k}{2p_k+1} - p_k + 1 \right)
\]

224
\[ \|\nabla u(t)\|_2^2 \leq b_0(b_k + 1)\|u(t)\|_2^{2(p_k+1)} \log|u_k| \]

According to the Poincare’s inequality, we have
\[ C_p \|u(t)\|_2^2 \leq \|\nabla u\|_2^2 \] for sufficiently small \(|\Omega|\), we have that
\[ \frac{\partial^2}{\partial t^2} \|u(t)\|_2^2 \leq \left(\frac{2(p_k+1)^2p_k}{2p_k+1} - p_k + 1\right) C_p \|u(t)\|_2^2 \leq b_0(b_k + 1) \|u(t)\|_2^{2(p_k+1)} \log|u_k| \]

1) \(\|u(t)\|_2^{p_k+1} \log|u|\)

Let \(\frac{2(p_k+1)p_k}{2p_k+1} < 1\), then
\[ \frac{\partial^2}{\partial t^2} \|u(t)\|_2^2 \leq \left(-\frac{2(p_k+1)p_k}{2p_k+1} + p_k + 1\right) \|u(t)\|_2^2 + b_0(b_k + 1)\|u(t)\|_2^{2(p_k+1)} \log|u_k| \]
\[ \frac{\partial^2}{\partial t^2} \|u(t)\|_2^2 + (p_k + 1) - \frac{2(p_k+1)^2p_k}{2p_k+1} \|u(t)\|_2 \leq b_0(b_k + 1)\|u(t)\|_2^{14} \log|u_k| \]

By young inequality, we obtain
\[ \frac{\partial^2}{\partial t^2} \|u(t)\|_2^2 + \|u(t)\|_2^2 \leq C \]

Where \(C\) is a constant independent of \(u\). By \(u_0, u_1\) are periodic control function, we get \(\|u(t)\|_2^2 \leq R\), where \(R\) is a positive constant.

**Theorem (2.2):** If (a),(b) and (c) hold, then the problem (1.1)-(1.3) admits at least one periodic solution \(u\).

Proof First, we define a map by considering the following problem:
\[ u_{tt} - \text{div}(\sigma(|\nabla u|^2) \nabla u^m) - (\nabla u)^p = |u|^{m-2}f(x, t, u) \log|u|, \quad x \in \Omega, \ t > 0, \]
\[ u(x, t) = 0, x \in \partial\Omega, \quad t \in [0, 1], \]
\[ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega, \]

Where \(f(x, t, u)\) is a given function in \(C_T(\overline{\Omega})\). If follows from a standard argument similar to [11] that the problem (1.1)-(1.3) admits a unique solution. So, we define a map \(T: C_T(\overline{\Omega}) \rightarrow C_T(\overline{\Omega})\). By \(u = Tf\) is compact and continuous. In fact, by the method in [15], we can infer that \(\|u\|_{L^\infty(\overline{\Omega})}\) is bounded if \(f \in L^\infty(\overline{\Omega})\) and \(u, \nabla u \in C^\alpha(\overline{\Omega})\). For some \(\alpha > 0\). Then by the Arzela-Ascoli theorem the compactness of the map \(T\) comes from \(\|u\|_{L^\infty(\overline{\Omega})}\) and Holder continuity of \(u\). The continuity of the map \(T\) comes from the Holder continuity of \(\nabla u\).

3. Conclusion:

In conclusion, this paper aimed to study the existence of solutions for certain types of hyperbolic differential equations with periodicity of certain control functions as nonlocal initial conditions. The techniques used in this study were based on analytic dependencies and made use of fixed point theorems to ensure the existence of solutions. This work represents a significant advancement in the field of hyperbolic differential equations and the application of nonlocal initial conditions.
4. References:


Article submitted 25 December 2022. Published as resubmitted by the authors 30 December 2022.