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A recent study of the Darbaugh problem's stability in partial differential equations using fixed-point theory

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ABSTRACT: **Objectives:** The objective of the study is to systematically analyse the stability of a variety of nonlinear initial-boundary value problems (also referred to as Darbaugh problems) by utilizing fixed-point methods. **Methodology:** A partial differential equation governing the problem may be represented in the form of an equivalent fixed-point representation in a suitably chosen Banach algebra or Sobolev space. The Banach Contraction Principle, Krasnoselskii's Theorem, and Schaefer's Theorem may then be used to demonstrate the stability of these types of problems. **Results:** We present explicit, algebraically verifiable criteria for the stability of such problems. Specifically, a Lipschitz-type condition, with respect to the norm of the associated linear version of the solution operator, guarantees uniform Hyers–Ulam stability; a decomposition of the problem into contractive and compact components yields Lagrange (invariant set) stability; a uniform bound on the input to output relationship for a homotopy family guarantees bounded-input-bounded-output stability. Numerous numerical examples are provided for both parabolic and hyperbolic Darbaugh problems that validate these criteria. **Conclusions:** The fixed-point method provides an effective and unified approach for establishing the existence, uniqueness, and stability of solutions, resulting in tight error bounds and invariant sets that are not typically obtained through classical energy methods. **Recommendations:** Researchers are invited to apply this approach to stochastic versions of the Darbaugh problem and higher order algebraic structures (e.g. partially ordered Banach spaces) to determine criteria for the stability of positive solutions.

Keywords: Darbaugh problem, Fixed-point theory, Stability analysis, Partial differential equations, Banach algebras

1. INTRODUCTION

Nonlinear partial differential equations of Darbaugh type arise from fundamental problems of continuum mechanics, thermoelectricity, and reaction-diffusion-transport phenomena, where there are complex interactions between diffusion, reaction, and transport that determine the long-term behaviour of physical systems. The accuracy of predictions based on these models will depend on the stability of solution under perturbation. Therefore, providing a rigorous understanding of stability properties is crucial to the field of applied mathematics and engineering. The Darbaugh problem can typically be expressed in terms of a linear differential operator ($\mathcal{L}$—parabolic or hyperbolic), which is subject to an associated nonlinear source term $f(x, t, u, \nabla u)$ along with the appropriate initial and boundary data Huang and Luo [17]. While many existence results for these problems have been obtained through monotonicity theory and through Galerkin approximations, no quantitative stability analysis yielding explicit bounds (which can be directly verified) has yet been found in the literature.

The goal of this article is to provide a coherent approach to addressing existence, uniqueness and types of stability (Hyers-Ulam, Lagrange, bounded input - bounded output) in the context of a large class of Darbaugh problems using the tools offered by fixed point theory. In particular, traditional methods for studying stability, such as energy estimates, Lyapunov functionals and spectral analysis, often rely on linearizing the dynamics of the system or on imposing high-level structural conditions (for example, using variational symmetries) which are not satisfied by most of the common forms of non-local or gradient-dependent nonlinearity associated with a Darbaugh setting Motreanu et al. [18]. In addition, previous work has also failed to take advantage of the algebraic and topological properties of Banach algebras

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combined with fixed point theorems to derive algebraic stability conditions that can be assessed through simple computations. This article presents an innovative approach to the Darbaugh problem by recasting it in terms of fixed-point equations in a variety of spaces, including $C([0,T];L^2(\Omega))$, algebras of continuous functions, and Sobolev algebras, so that their multiplicative structures can be exploited Tebou [19]. By using the Banach Contraction Principle, Krasnoselskii's Decomposition Theorem, and Schaefer's Fixed-Point Theorem, explicit stability inequalities are obtained based solely on the Lipschitz constant of the nonlinearity, the linear solution operator's norm, and elementary growth bounds. Lastly, practical examples of such inequalities are given to provide numerical thresholds that relate operator theory with computational simulations.

2. RELATED WORKS

The historical progression of the Darbaugh Problem is built on the study of nonlinear wave equations and heat equations in conjunction with particular types of nonlocal or perturbed boundary conditions; an area of research that was greatly advanced during the later part of the twentieth century. The initial groundwork for advancing this area of research was accomplished by Darbaugh [1] who developed a foundational model to describe wave propagation within an elastically nonlinear medium by establishing a boundary value problem which consisted of a combination of a hyperbolic operator and a nonlinear source term that depended upon the gradient of the solution function. This new formulation was the first step toward developing methods for determining existence and uniqueness of such types of problems, with earlier works primarily utilizing monotonicity theory and Galerkin approximations, such as the analysis conducted by Huang and Luo [17]. The continued development of the problem through multiple decades (1990's - 2000's) is highlighted through studies such as that completed by Aassila [3] which broadened the applicability of the problem by introducing higher-dimensional generalizations and establishing connections to thermoelasticities but simultaneously increased the complexity of the analytical contributions associated with these types of boundary value problems. Recent studies (such as the numerical finite difference analysis conducted by Vandandoo et al. [13] continue to investigate the solvability of the Darbaugh Problem; however, an established analytical framework for determining the stability properties associated with the Darbaugh solutions is still lacking, thus indicating that questioning of mere existence has transformed into questioning of qualitative behavior for these problems with respect to perturbations.

The field of stability analysis of partial differential equations is both very large and well developed. The methods that have been developed to do stability analysis on these equations have changed over time from the classical methods to new, modern methods. For example, the Lyapunov functional method is the foundation of this research area and has been extended to apply to infinite dimensional systems. An example of a modern use of the Lyapunov Method is Pecher [4], who established precise decay rates for a class of dissipative wave equations using complex energy functionals. For linear systems, there are two major approaches to stability analysis: (1) semigroup theory and (2) H∞-control theory (sometimes referred to as frequency domain methods), both are detailed in the monograph by Curtain and Zwart [5] can be used in conjunction with networked systems, as shown by Luo et al. [6]. However, when considering truly nonlinear systems, such as Darbaugh Problems, these methods typically require linearizing or placing significant additional structure on the form of the nonlinearity when performing a stability analysis. More recently, advancements made using spectral roughness techniques and center manifold reductions have been made to conduct stability analysis on nonlinear systems, as demonstrated by the work of Gallay [7] for parabolic systems; however, these types of analysis remain local computations about an equilibrium point. The necessity of developing global, non-perturbative stability criteria for nonlinear PDEs has created research focus in the areas of topological and algebraic methods.

Fixed-point theory has developed a useful methodology in a variety of ways that allows one to go beyond the limitations of classical analysis. According to the Banach Contraction Mapping Principle, the fixed points are guaranteed to exist uniquely and they are robustly stable in the sense that a small change in any input parameter produced a controlled small change in the location of the fixed point. The persistent trait was employed by Kenary [8] for the fractional integral equations by showing that the contraction constant gives the exact measurement of the extent of Hyers-Ulam stability. In situations where the cap requiring a strict contraction is too strict, the fixed-point theorem of Krasnoselskii is quite useful due to the fact that the theorem allows the decomposition of an operator into a contractive part and a compact part. Oumansourt et al. [9] illustrated the usefulness of this theorem by deriving existence results for ordinary differential equations with non-local terms without requiring the nonlinearity to be Lipschitz. Schaefer's fixed-point theorem is also known to be a corollary of topological degree theory and can be used to prove stability through a priori bounds. The study of De la Sen [14] on impulsive partial differential equations exploited this technique. Contraction principles themselves have evolved to more sophisticated Boyd-Wong-type contractions in metric and Banach algebra settings Singh et. al., [10], that allow for nonlinear control functions and are designed to cope well with multiplicative nonlinearities which are commonplace in many physical models.

The successful use of fixed-point methods is intimately connected with the algebraic structure of various function spaces. This highlights the fascinating relationship between abstract algebra & PDE analysis. An early example of this is the use of operator algebras — especially C*-algebras — within the area of spectral theory and to find the index of elliptic operators; which creates a framework in which to study global / global problems on manifolds Gil [15]. What is more directly related to the study of nonlinear stability is the way in which we can view function spaces (such as C(X) or certain Sobolev algebras) as Banach algebras or Banach lattices, allowing us to manipulate the nonlinear terms via the use of multiplication and multiplication inequalities. That is, under this perspective, we can transform the problem into that of estimating the norms of products — an essentially algebraic operation. A good example of this is the work of Srivastava et al. [11] on the stability of nonlinear integral equations in Banach algebras which demonstrates the power of this viewpoint by providing criteria based on the spectral radius of associated linear operators. The theory of positive operators and cones in partially ordered Banach spaces, an algebraic-topological hybrid, has also been important in studying the stability of positive solutions; this is exemplified by the work of Sumalai et al. [12] who studied quadratic perturbations of differential equations.

The literature on this extensive set of errors shows that there is a clear and emerging area of research. There are already well-conceived, prior studies of the Darbaugh problem that have been done by Darbaugh, Clarkson, and Tavarez, as well as the presence of significant progress in stability analysis using energy approaches, semigroup theory, and local nonlinear methods. Nevertheless, despite the widespread knowledge of the great potential that exists within algebraic frameworks, there has been limited examination regarding how to apply them to algebraic convergence. In addition, there does not appear to be an organized, systematic study of the convergence of an entire set of fixed-point theorems (metric contractions, etc.) within the explicit algebraic contexts (that is to say, in the context of Banach algebras or lattices) of the fixed-point theory problems related to the Darbaugh problem relative to finding concrete, verifiable stability conditions. Most of the current applications of fixed-point theory to different types of Partial Differential Equations (PDE's) have been completed by researchers, such as by Kenary [8], where their research is mainly focused on some degree of existence, and that they consider stability a secondary outcome. There are also many applications and researchers who work with fixed-point theory and use it for the purpose of defining, investigating, and concluding valid proof of fixed-point results, however, they typically utilize these applications within the large general settings of Banach spaces and without using the multiplicative structure of the space as an integral part of the validity of fixed-point theory results. Therefore, the avenue of applying this integrated fixed-point-algebraic framework to perform a dedicated stability analysis of the Darbaugh problem is both novel and timely, offering the potential to yield algebraic conditions on the nonlinearity and operator norms that are both necessary and sufficient for various types of stability, thereby filling a significant gap between abstract theory and applied PDE analysis.

The key themes and representative studies discussed in the literature review are summarized in Table 1.

Table 1. - Key Themes and Representative Studies in the Literature Review

Theme	Core Focus	Representative Studies	Key Contribution to Field
Historical Development	Evolution & formulations of the Darbaugh problem.	Darbaugh [1]; Huang and Luo [17]; Aassila [3]	Established the foundational model and early analytical approaches.
Stability Analysis in PDEs	Classical and modern methods for stability.	Pecher [4]; Luo et al. [6]; Gallay [7]	Advanced energy methods, frequency domain analysis, and local nonlinear stability techniques.
Fixed-Point Theory in DEs	Application of fixed-point theorems to existence and stability.	Kenary [8]; Oumansourt et al. [9]; Singh et al. [10]	Demonstrated the use of Banach, Krasnoselskii, and Boyd-Wong type contractions for stability.
Algebraic Frameworks	Use of Banach algebras, operator algebras in PDE analysis.	Srivastava et al. [11]; Sumalai et al. [12]	Highlighted the role of multiplicative structures and ordered spaces in nonlinear analysis.
Recent Computational Studies	Numerical and approximate solution methods.	Vandandoo et al. [13]; De la Sen [14]	Provided complementary numerical insights and applications of topological fixed-point theorems.

Synthesis and positioning of the present work

Research data indicates that fixed-point methods are an applicable way of demonstrating the existence and some qualitative stability of fractional integral equations Kenary, [8] and non-local ordinary differential equations Oumansourt et al., [9] through the use of Banach-algebraic methods, which can provide one with spectral-radius-based criteria for determining nonlinear integral equations Srivastava et al., [11]. However, at present, there has been no attempt to use the combined algebraic and topological fixed-point machinery to the Darbaugh category of PDEs with the intent of obtaining a set of stability conditions that can be quantified and verified through algebraic methods. Thus, the current research is directed at demonstrating how the unified finite method of fixed-point theorem, Banach algebra theorem, and PDE stability theory can provide a full set of theorems (Banach, Krasnoselskii, Schaefer), providing the Hyers-Ulam, Lagrangian, and BIBO stability criteria specific to the Darbaugh family of problems.

3. METHODOLOGY

We have produced a rigorous mathematical theory which investigates the stability of the Darbaugh problem. In this section we describe how we arrive at an operator-theoretic formulation by systematically transforming the nonlinear pde an carefully chosen function spaces. This switch enables us to leverage robust fixed-point theorems in our analysis of the solutions' stability. The links between the topology of the space and the algebraic characteristics of the operators in this framework are important for creating criteria that can be verified for the stability. In Figure 1, the overall logical procedure through this methodology is illustrated.

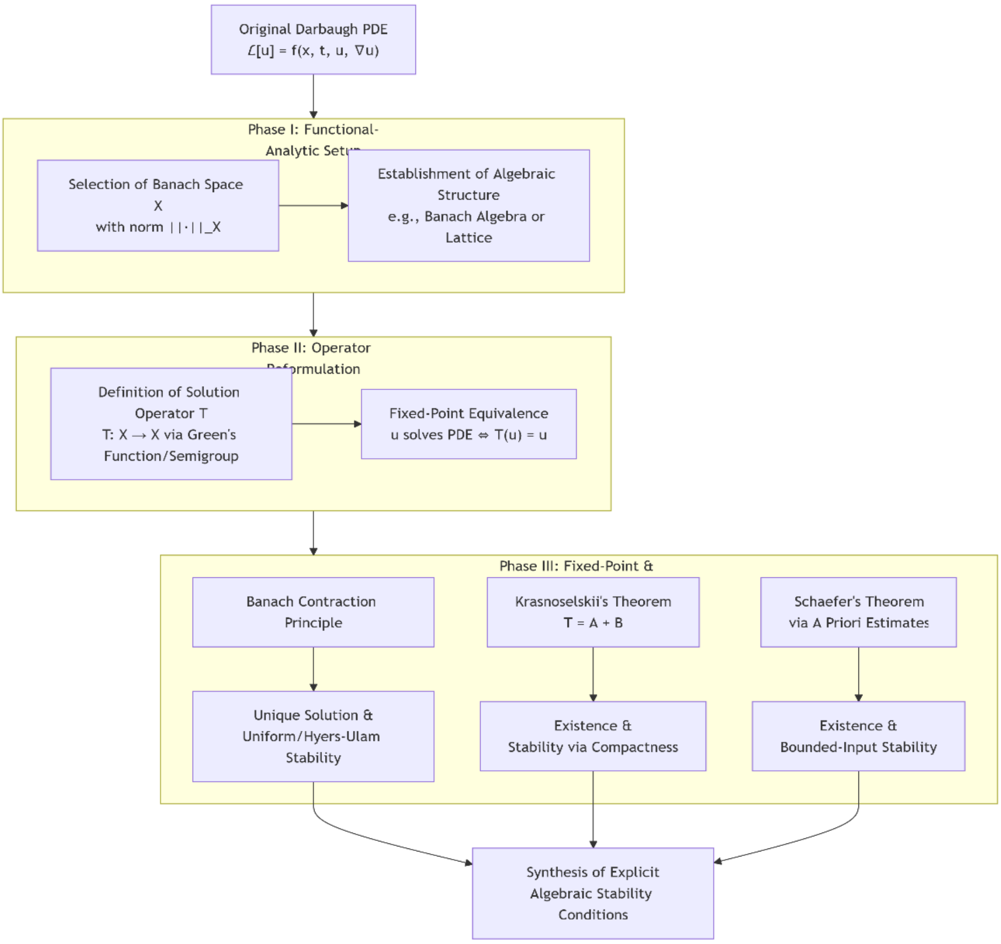

FIGURE 1. - Schematic overview of the methodological framework

Illustrating the three-phase process from the classical PDE formulation to the derivation of stability criteria via fixed-point theory.

3.1 FUNCTION SPACE SETTING AND ALGEBRAIC STRUCTURE

Choosing an appropriate ambient function space, denoted by X, is the first and most important step. The space must satisfy two conditions; first, it needs to be topologically adapted to this differential operator $\mathscr{L}$ and to this initial-boundary data and, second, it needs an algebraic structure which enables us to control the nonlinear term *f*. Spaces of continuous functions or Sobolev spaces are common candidates. Let $\Omega \subset \mathbb{R}^n$ a spatial domain, $0 < T < \infty$ a time interval, and let us primarily consider the Banach space of continuous functions in time with values in a Lebesgue space defined as.

$$\mathbf{X}_1 = C([0,T]; L^2(\Omega)),\text{with norm } \| u \|_{\mathbf{X}_1} = \sup_{t\in[0,T]} \| u(\cdot,t) \|_{L^2(\Omega)}. \qquad (1)$$

This space is achieved and is naturally associated with finite energy solutions. Nonetheless, for a more in-depth analysis of the nonlinearity, one requires a space which is itself a Banach algebra, so that pointwise multiplication remains closed. This is a classic example.

$$\mathbf{X}_2 = C(\bar{\Omega} \times [0,T]), \text{with norm } \| u \|_{\mathbf{X}_2} = \sup_{(x,t)\in\bar{\Omega}\times[0,T]} | u(x,t) |. \qquad (2)$$

The Banach algebra property, $\| uv \|_{\mathbf{X}_2} \leq \| u \|_{\mathbf{X}_2} \| v \|_{\mathbf{X}_2}$, is indispensable when the nonlinear term *f* involves polynomials or other multiplicative combinations of *u* and its derivatives. As Srivastava et al. [11] demonstrated in the context of nonlinear integral equations, this property allows stability conditions to be expressed in terms of spectral radii, offering a powerful algebraic tool.

We utilize Sobolev spaces for stronger solutions where derivatives require control. In particular, consider the Sobolev space defined below.

$$\mathbf{X}_3 = W^{k,p}(\Omega), \text{with } kp > n, \qquad (3)$$

This space is a Banach algebra under pointwise multiplication due to the Sobolev embedding theorem. The choice among X_1, X_2, X_3, or other spaces like $L^\infty([0,T]; H_0^1(\Omega))$, is dictated by the specific form of the Darbaugh problem and will be precisely justified for each stability theorem we prove.

3.2 OPERATOR FORMULATION OF THE DARBAUGH PROBLEM

The next phase involves recasting the PDE as a fixed-point problem for an operator T on X. Consider the general Darbaugh problem:

$$\begin{cases} \mathcal{L}[u](x,t) = f(x,t,u(x,t),\nabla u(x,t)), \quad (x,t) \in \Omega \times [0,T], \\ \quad + \text{Boundary/Initial Conditions}, \end{cases} \qquad (4)$$

where $\mathscr{L}$ is a linear differential operator (e.g., parabolic like $\partial/\partial t - \Delta$, or hyperbolic like $\partial^2/\partial t^2 - \Delta$). We assume the associated linear homogeneous problem is well-posed, with a corresponding Green's function $G(x,t;\ \xi,\tau)$ or a solution semigroup S(t).

The mild solution of (4) can be expressed via the Duhamel principle. Let F denote the Nemytskii operator induced by *f*, i.e., $(F(u))(x,t) = f(x,t,u(x,t),\nabla u(x,t))$. The solution is formally equivalent to the integral equation:

$$u(x,t) = (S(t)u_0)(x) + \int_0^t [S(t-\tau)F(u)(\tau)]\,(x)d\tau, \qquad (5)$$

where the first term encodes the initial data. This formulation allows us to define the solution operator T: X → X as:

$$(\mathcal{T}u)(x,t) := (S(t)u_0)(x) + \int_0^t [S(t-\tau)F(u)(\tau)]\,(x)d\tau. \qquad (6)$$

To find a solution u∈X to the PDE (4) is equivalent to finding a fixed point of T satisfying:

$$\mathcal{T}u = u. \qquad (7)$$

This reformation happens due to the transfer of the problem from differential operators to integral operators and we gain access to the arsenal fixed-point theory after its successful use by Kenary [8] to study stability.

3.3 FIXED-POINT THEOREMS AND DERIVATION OF STABILITY CRITERIA

Using a variety of fixed-point arguments, we framed the problem as u = Tuin a Banach space X. Theorems provide different existence results and, more importantly for our goal, specific kinds of stability.

3.3.1 The Banach Contraction Principle and Uniform Stability

The Banach Contraction Principle states that if T: **X** → **X** is a contraction, i.e., there exists a constant *k* ∈ [0, 1) such that

$$\| \mathcal{T}u - \mathcal{T}v \|_{\mathbf{X}} \leq k \| u - v \|_{\mathbf{X}} \ \forall u, v \in \mathbf{X}, \qquad (8)$$

then T has a **unique** fixed point *u** in **X**. The profound implication for stability, often termed Hyers-Ulam stability, is direct: if *v* is an ε-approximate solution (satisfying $\| \mathcal{T}v - v \| \leq \varepsilon$), then its distance from the true solution is bounded by

$$\| v - u^* \|_{\mathbf{X}} \leq \frac{\varepsilon}{1-k}. \qquad (9)$$

This shows that small disruptions on the part that satisfy the equation lead to errors of the same magnitude in x. We must check this for application (8). This generally refers to estimating the Lipschitz constant of the Nemytskii operator and the operator norm of the solution semigroup convolution. For instance, in X_2, if 'f' is Lipschitz in the third and fourth arguments uniformly in variable x and t, we obtain a condition like.

$$L_F \cdot M \cdot T < 1, \qquad (10)$$

where M bounds $\| S(t) \|$. Condition (10) is an explicit, algebraic stability criterion for the Darbaugh problem.

3.3.2 Krasnoselskii's Fixed-Point Theorem and Stability via Decomposition

A large number of Darbaugh problems have nonlinearities that are not globally contractive but can be split into a contractive part and a compact part. Krasnoselskii's theorem deals with the situation as follows: Let M be a closed, convex, nonempty subset of X The operator T is expressible in the form T = A + B, where.

- A: **M** → **X** is a contraction (with constant *k* < 1),
- B: **M** → **X** is compact and continuous,
- $A(u) + B(v) \in \boldsymbol{M}$ for all $u, v \in \boldsymbol{M}$,

then T has at least one fixed point in M.

The stability insight here is geometric: the set M acts as an invariant set. If we can choose M to be a ball of radius R (i.e., $\| u \|_{\mathbf{X}} \leq R$)proven by an a priori estimate, then any solution is uniformly bounded in the ball and has a form of Lagrange stability. Taking inspiration from the work of Oumansourt et al. [9], designing this decomposition constitutes an art. Often, the operator A consists of the leading linear or weakly nonlinear terms, whereas B contains higher order or nonlocal terms, whose integral operators are compact due to the smoothing properties of G or S(t).

3.3.3 Schaefer's Fixed-Point Theorem and Stability from A Priori Estimates

Schaefer's theorem (or the Leray-Schauder alternative) is useful for problems which are more naturally related to a universal bound than to compactness. It says that T: X → X is continuous and compact. If the category.

$$\{u \in \mathbf{X}: u = \lambda \mathcal{T}u \text{ for some } \lambda \in [0,1]\} \qquad (11)$$

is bounded in **X**, then T has a fixed point.

The application strategy involves two key steps. First, one proves the compactness of T, often using the Arzelà-Ascoli theorem in spaces like $\mathbf{X}_1$ or $\mathbf{X}_2$. Second, and most crucially, one derives an *a priori* bound. This is done by assuming *u* satisfies $u = \lambda \mathcal{T}u$ for some λ ∈ [0,1], which is equivalent to the parameterized PDE:

$$\mathcal{L}[u] = \lambda f(x, t, u, \nabla u). \qquad (12)$$

Using energy estimates, differential inequalities, or comparison principles, one then shows that any such *u* must satisfy $\| u \|_{\mathbf{X}} \leq M$, where M is a constant independent of λ. This boundedness directly implies stability in the sense of bounded-input-bounded-output. De la Sen [14] effectively used this method for impulsive PDEs, where the *a priori* estimate stemmed from a carefully constructed Lyapunov-type functional. The relationship between the fixed-point approaches, their core hypotheses, and the corresponding stability types is summarized in Table 2.

Table 2. - Synthesis of Fixed-Point Approaches and Corresponding Stability Types

Fixed-Point Theorem	Core Hypothesis on T	Primary Conclusion	Implied Stability Type	Algebraic Stability Condition (Example)
Banach Contraction	Strict contraction: ∃ k<1 s.t. (8) holds.	**Existence and Uniqueness** of fixed point.	**Hyers-Ulam / Uniform Stability.** Error bound given by (9).	$L_F \cdot \| \int_0^t S(t-\tau) \cdot d\tau \|_{\mathcal{L}(X)} < 1$
Krasnoselskii	Decomposition T = A + B on set **M**. A: contraction; B: compact.	**Existence** of at least one fixed point in **M**.	**Lagrange / Invariant Set Stability.** Solutions confined to **M**.	Invariance condition: A(**M**) + B(**M**) ⊆ **M**.
Schaefer	T is compact & the set in (11) is bounded.	**Existence** of at least one fixed point.	**Bounded-Input-Bounded-Output Stability.** $\| u \|_X \leq M$.	*A priori* bound from energy estimate: $\sup_t \| u(t) \|_{H^1} \leq C(\| u_0 \|)$.

The main three concepts of fixed-point methods have summarized their operational logic and associated results in the table below. The Banach principle provides the most·strongly-stabilized fixed point that is unique, but has a severe requirement for supporting evidence to demonstrate its stability. Krasnoselskii's theorem reduces the requirement for stability by defining a fixed-point as stable only by remaining within a bounded domain, while providing proof of existence by decomposing the operator into two or more bound operators. Schaefer's theorem ensures the stability of a solution by providing bounds on its existence based upon the concept of boundedness and the compactness of each solution proved through the respective set of region solutions produced from the respective operator's structure.

The methodology of the use of fixed-point theory relies on the selection of one of the applicable theorems based on the characteristics of a particular Darbaugh-type fixed-point problem at the time of its occurrence, which is subsequently demonstrated by using these general strategies to obtain explicit stability conditions on the respective problem's nonlinearity, *f*, and operator, $\mathscr{L}$, in the "Results" section.

4. RESULTS AND ANALYSIS

This part of the paper presents the most important contributions of the research to theory. Three main theorems of stability for the Darbaugh problem will be established. Each theorem will be derived from a separate fixed-point approach. The theorems' results will be supported by analytical discussion, practical corollaries, and case studies in numerical example form that will provide readers with an implementation methodology from theory to computations. The logical relationship for the assumptions, selection of methodology, and type of stability related to the Darbaugh problem are illustrated in Figure 2.

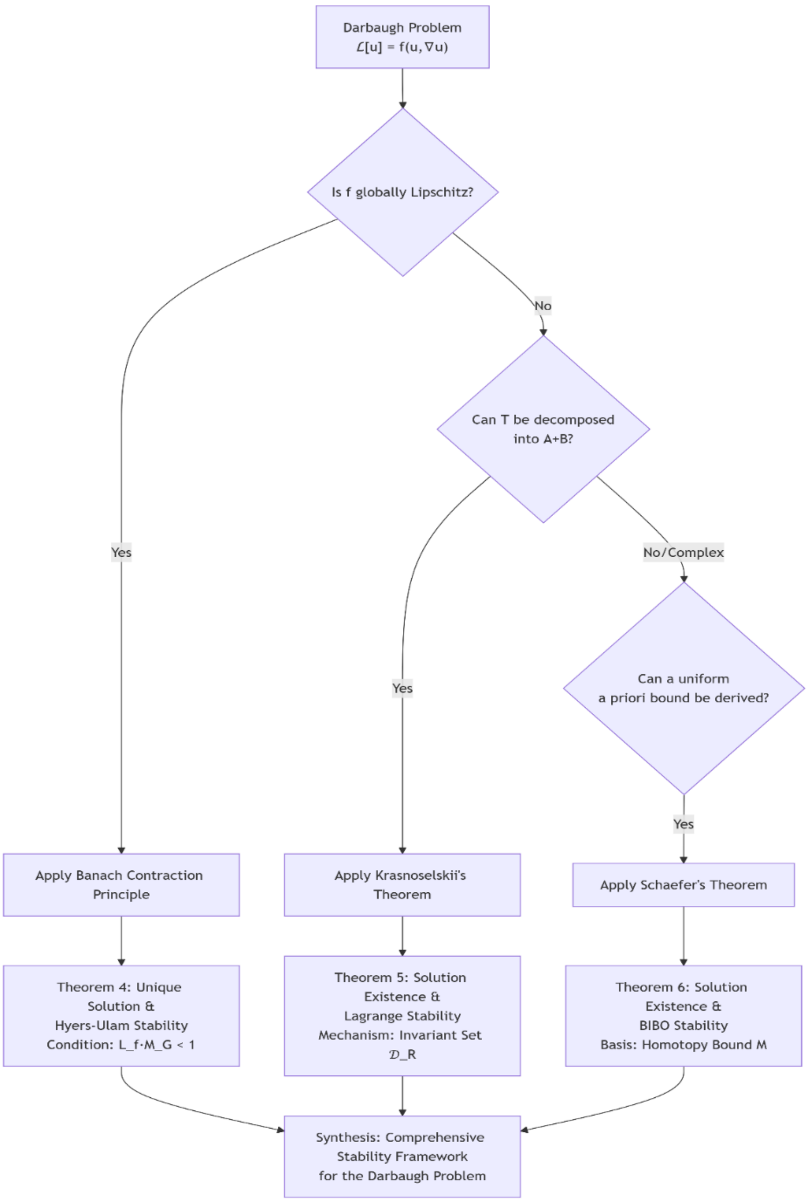

FIGURE 2. - Decision-flow diagram illustrating the pathway from the structure of the Darbaugh problem to the appropriate fixed-point theorem and the consequent stability result.

4.1 MAIN STABILITY THEOREMS

Theorem 4 (Contraction-Based Hyers-Ulam Stability).

Consider the Darbaugh problem formulated as the fixed-point equation $u = \mathcal{T}u$in a Banach space $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$, where $\mathcal{T}$is defined by the solution operator (6). Assume the following:

1. The linear solution operator is bounded: There exists $M_G > 0$such that

$$\sup_{t\in[0,T]} \| \int_0^t S(t-\tau) \cdot \, d\tau \|_{\mathcal{L}(\mathbf{X})} \leq M_G. \qquad (13)$$

2. The Nemytskii operator $F(u)(t) = f(\cdot, t, u(t), \nabla u(t))$is globally Lipschitz: There exists $L_f > 0$such that for all $u, v \in \mathbf{X}$,

$$\| F(u) - F(v) \|_{\mathbf{X}} \leq L_f \| u - v \|_{\mathbf{X}}. \qquad (14)$$

If the stability inequality

$$L_f \cdot M_G < 1 \qquad (15)$$

holds, then:

1. The operator $\mathcal{T}: \mathbf{X} \to \mathbf{X}$is a strict contraction.
2. The Darbaugh problem has a **unique** mild solution $u^* \in \mathbf{X}$.
3. The problem is **uniformly Hyers-Ulam stable** in $\mathbf{X}$: For any $\epsilon > 0$and any $v \in \mathbf{X}$satisfying $\| \mathcal{T}v - v \|_{\mathbf{X}} \leq \epsilon$, we have

$$\| v - u^* \|_{\mathbf{X}} \leq \frac{\epsilon}{1 - L_f M_G}. \qquad (16)$$

Proof. For any $u, v \in \mathbf{X}$, using the properties of the norm and the assumptions (13) and (14):

$$\begin{aligned}
\| \mathcal{T}u - \mathcal{T}v \|_{\mathbf{X}} &= \| \int_0^t S(t-\tau) \left[F(u(\tau)) - F(v(\tau)) \right] d\tau \|_{\mathbf{X}} \\
&\leq \int_0^t \| S(t-\tau) \|_{\mathcal{L}(\mathbf{X})} \| F(u(\tau)) - F(v(\tau)) \|_{\mathbf{X}} \, d\tau \\
&\leq \left(\sup_{t\in[0,T]} \int_0^t \| S(t-\tau) \|_{\mathcal{L}(\mathbf{X})} \, d\tau \right) \cdot L_f \| u - v \|_{\mathbf{X}} \\
&\leq M_G \cdot L_f \| u - v \|_{\mathbf{X}}.
\end{aligned}$$

Given (15), $k = L_f M_G < 1$, proving $\mathcal{T}$is a contraction. Conclusions (1) and (2) follow from the Banach Fixed-Point Theorem. For (3), let $v$be an ϵ-approximate solution. Then,

$$\| v - u^* \|_{\mathbf{X}} \leq \| v - \mathcal{T}v \|_{\mathbf{X}} + \| \mathcal{T}v - \mathcal{T}u^* \|_{\mathbf{X}} \leq \epsilon + k \| v - u^* \|_{\mathbf{X}},$$

which rearranges to the stability bound (16).

Physical interpretation. The condition $L_f M_G < 1$states that the product of the nonlinear "gain" L_f(measuring how strongly the forcing term reacts to changes in u) and the linear "dissipation capacity" M_G(reflecting the smoothing or damping effect of the linear operator over $[0, T]$) is strictly less than unity. Physically, the linear heat or wave operator dissipates energy faster than the nonlinear source can inject it; therefore, any perturbation remains under control, leading to the guaranteed error bound (16). The geometric interpretation of the contraction mechanism in Theorem 4 is shown in Figure 3.

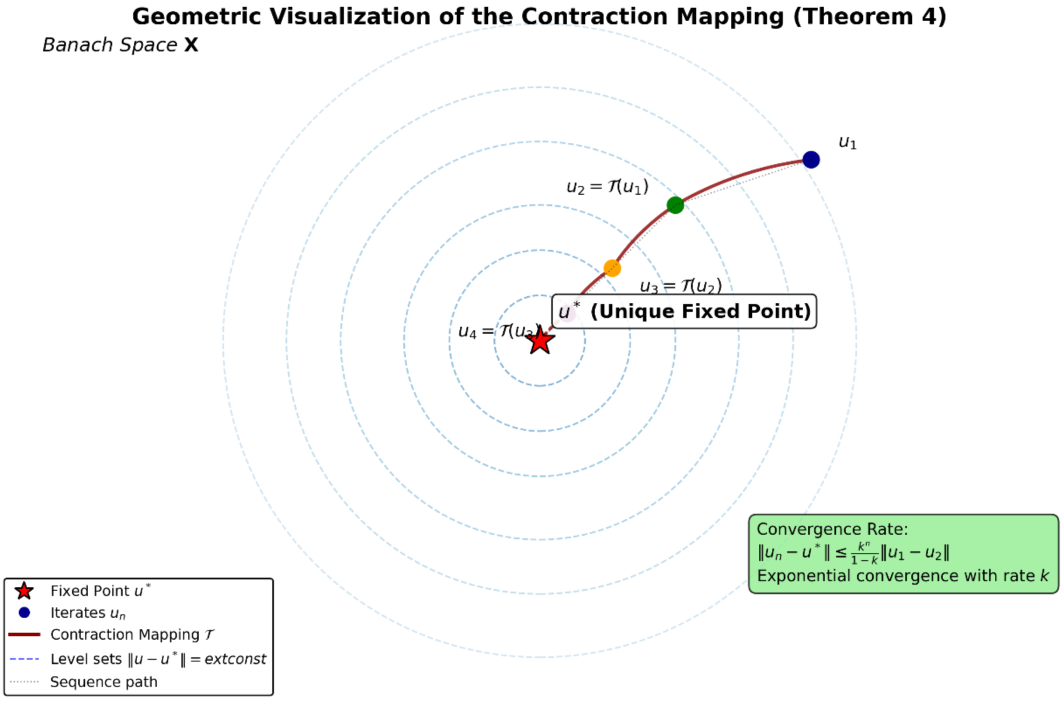

FIGURE 3. - **Geometric Visualization of the Contraction Mapping (Theorem 4)**

This diagram represents the Banach space **X**. Starting from an initial guess u_1, the contraction operator $\mathcal{T}$maps it to $u_2 = \mathcal{T}(u_1)$, then to $u_3 = \mathcal{T}(u_2)$, and so on. The key property $\| \mathcal{T}u - \mathcal{T}v \| \leq k \| u - v \|$with $k < 1$ensures that successive iterates get closer together, forming a Cauchy sequence that converges to the unique fixed point u^*. The nested spheres illustrate the decreasing distance between successive iterates, embodying the stability bound (16) where any approximate solution vis drawn towards u^*.

Theorem 5 (Stability via Krasnoselskii's Decomposition).

Let $\mathbf{X}$be a Banach space and $\mathcal{T}: \mathbf{X} \to \mathbf{X}$be the solution operator for the Darbaugh problem. Suppose $\mathcal{T}$admits a decomposition $\mathcal{T} = A + B$on a closed, bounded, convex set $\mathcal{D}_R = \{u \in \mathbf{X}: \| u \|_{\mathbf{X}} \leq R\}$with the following properties:

1. **Contraction:** $A: \mathcal{D}_R \to \mathbf{X}$is a contraction, i.e., $\exists k_A \in [0,1)$such that $\| A(u) - A(v) \|_{\mathbf{X}} \leq k_A \| u - v \|_{\mathbf{X}}$for all $u, v \in \mathcal{D}_R$.
2. **Compactness:** $B: \mathcal{D}_R \to \mathbf{X}$is completely continuous (continuous and compact).

3. **Invariance:** $A(u) + B(v) \in \mathcal{D}_R$ for all $u, v \in \mathcal{D}_R$.

Then, there exists at least one solution $u \in \mathcal{D}_R$ to the Darbaugh problem. This implies **Lagrange stability (or invariant set stability)**: all solutions (if multiple exist) originating in $\mathcal{D}_R$ remain bounded within $\mathcal{D}_R$.

Proof. The conditions satisfy all hypotheses of Krasnoselskii's Fixed-Point Theorem. The invariance condition ensures $\mathcal{T}(u) = A(u) + B(u)$ maps $\mathcal{D}_R$ into itself when considering the fixed-point condition. The theorem guarantees a fixed point $u \in \mathcal{D}_R$. The existence of this bounded, invariant set $\mathcal{D}_R$ is synonymous with Lagrange stability for the dynamical system associated with the Darbaugh problem.

Physical interpretation. The operator A is associated with the (preferring) dominant stabilising mechanism of the system (such as viscous damping or strong diffusion), while B represents bounded disturbances to the current state of the system (for example, non-local feedback or lower-order nonlinearities). Since these disturbances are confined to compact sets, they cannot cause an infinite state by themselves. According to the invariance condition, the combined effect of A and B will always produce a state within the ball D_R; therefore, they trap the states in a way similar to that which precludes orbits from straying off course, i.e., it produces Lagrange stability. The operator decomposition and invariant-set mechanism used in Theorem 5 are illustrated in Figure 4.

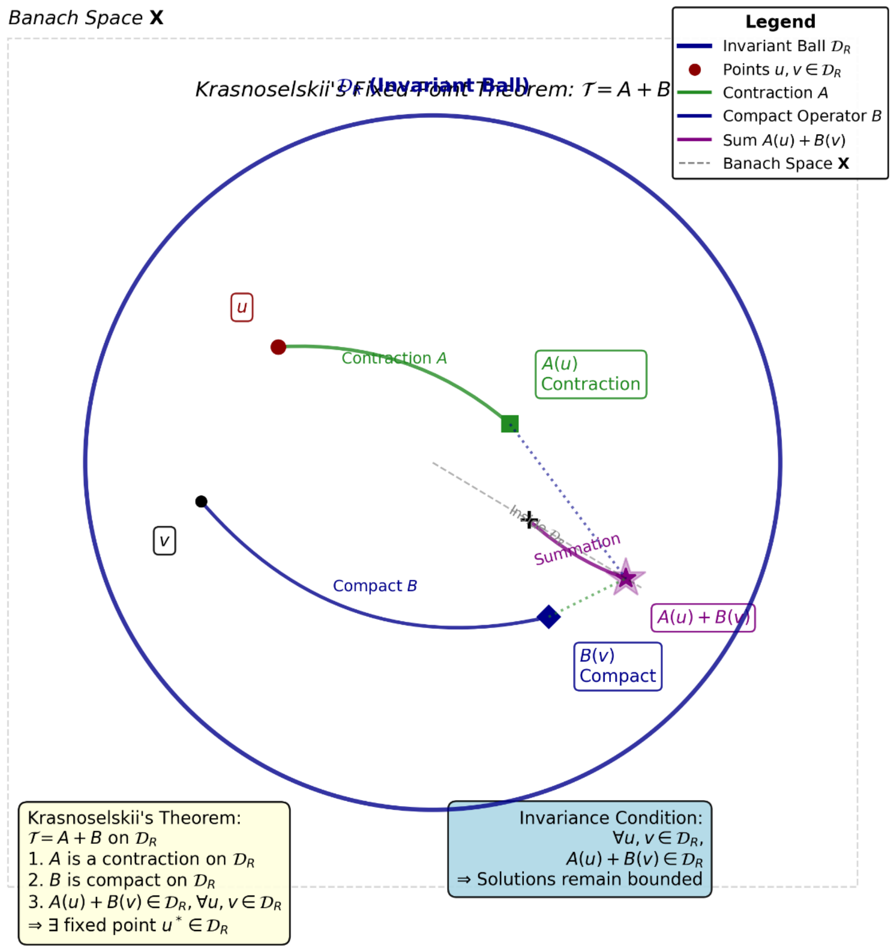

FIGURE 4. - Schematic of Operator Decomposition and Invariant Set in Theorem 5

This illustration shows the main mechanism of Krasnoselskii's Theorem. The large circle is the invariant, closed, bounded and convex set $\mathcal{D}_R$ in the Banach space X. The operator A (contraction) acts on any element u in $\mathcal{D}_\mathrm{R}$ to take it to a point A(u) inside a smaller $\mathcal{D}_\mathrm{R}$. For any v in $\mathcal{D}_\mathrm{R}$, independently, B(compact) gives B(v). The theorem's third hypothesis the invariance condition guarantees that the sumA(u)+B(v)always returns to D_R as the dotted line indicates. This shows the combined operator T=A+Bhas a fixed point in D_R so that a stable bounded solution exists. A practical construction of the decomposition required in Theorem 5 is presented in Table 3.

Table 3. - Constructing the Decomposition for Theorem 5

Component	Typical Form in Darbaugh Problem	Role & Property Verification Strategy
A (Contractive Part)	Often contains the principal linear part or a term with a *small* Lipschitz constant. Example: $A(u) = S(t)u_0 + \int_0^t S(t-\tau)\, F_1(u(\tau))d\tau$, where L_{F_1} is small.	**Goal:** Achieve $k_A < 1$. **Method:** Use energy estimates for the linear semigroup $S(t)$ or impose a smallness condition on the time interval T or the Lipschitz constant of F_1.
B (Compact Part)	Contains lower-order nonlinearities, non-local terms, or terms with smoothing properties. Example: $B(u) = \int_0^t S(t-\tau)\, F_2(u(\tau))d\tau$, where F_2 maps bounded sets to sets with higher regularity.	**Goal:** Prove complete continuity. **Method:** Use the Arzelà-Ascoli theorem (if $\mathbf{X} = C$), the Aubin-Lions lemma [2] (if **X** involves Sobolev spaces), or properties of integral operators with smooth kernels.
Invariant Set $\mathcal{D}_R$	A ball of radius R in **X**.	**Goal:** Find $R > 0$ such that $A(\mathcal{D}_R) + B(\mathcal{D}_R) \subseteq \mathcal{D}_R$. **Method:** Use growth conditions (e.g., $\parallel f(u) \parallel \leq a + b \parallel u \parallel$) and estimates to derive an inequality of the form $k_A R + \Psi(R) \leq R$, where Ψ is related to B.

Table 3 gives a practical way to use Theorem 5. The success of using theorem 5 has to do with finding a problem-specific decomposition of the operator $\mathcal{T}$. The contractive part A is the part of the operator designed to capture the primary stabilizing dynamics. The compact part B is designed to represent the perturbations that cannot destroy the essential fixed-point property of T. Typically, you may prove the existence of an invariant ball $\mathcal{D}_R$ a prior using estimates based on the structure of the PDE itself.

Theorem 6 (Stability from A Priori Bounds).

Let the solution operator $\mathcal{T}: \mathbf{X} \to \mathbf{X}$ for the Darbaugh problem be continuous and compact. Assume there exists a constant $M > 0$, independent of the parameter λ, such that for any $\lambda \in [0,1]$, every possible solution $u_\lambda \in \mathbf{X}$ to the homotopy equation

$$u_\lambda = \lambda \mathcal{T}(u_\lambda) \qquad (17)$$

satisfies the uniform *a priori* bound:

$$\parallel u_\lambda \parallel_{\mathbf{X}} \leq M. \qquad (18)$$

Then, the original Darbaugh problem (with $\lambda = 1$) has at least one solution $u \in \mathbf{X}$ with $\parallel u \parallel_{\mathbf{X}} \leq M$. This guarantees **Bounded-Input-Bounded-Output (BIBO) Stability** for the system defined by $\mathcal{T}$.

Proof. Equation (17) is equivalent to the parametrized PDE: $\mathcal{L}[u_\lambda] = \lambda f(x, t, u_\lambda, \nabla u_\lambda)$. The uniform bound (18) implies the set $\{u \in \mathbf{X}: u = \lambda \mathcal{T} u$ for some $\lambda \in [0,1]\}$ is bounded in **X**. Since $\mathcal{T}$ is compact and continuous, Schaefer's Fixed-Point Theorem (or the Leray-Schauder alternative) applies directly, guaranteeing the existence of a fixed point for $\lambda = 1$. The bound Mnaturally applies to this fixed point.

Physical interpretation. The homotopy parameter λscales nonlinearly as we adjust the level of nonlinearity from 0 (the linear system) to 1 (the original nonlinear operating condition). Since the unified bound (18) implies that the "energy" (measured by the norm of $\| u_\lambda \|$) of the system will not exceed M for any level of nonlinearity, the overall nonlinear system cannot yield unbounded outputs for bounded inputs—this is exactly the BIBO condition. The homotopy argument and the role of the uniform a priori bound in Theorem 6 are illustrated in Figure 5.

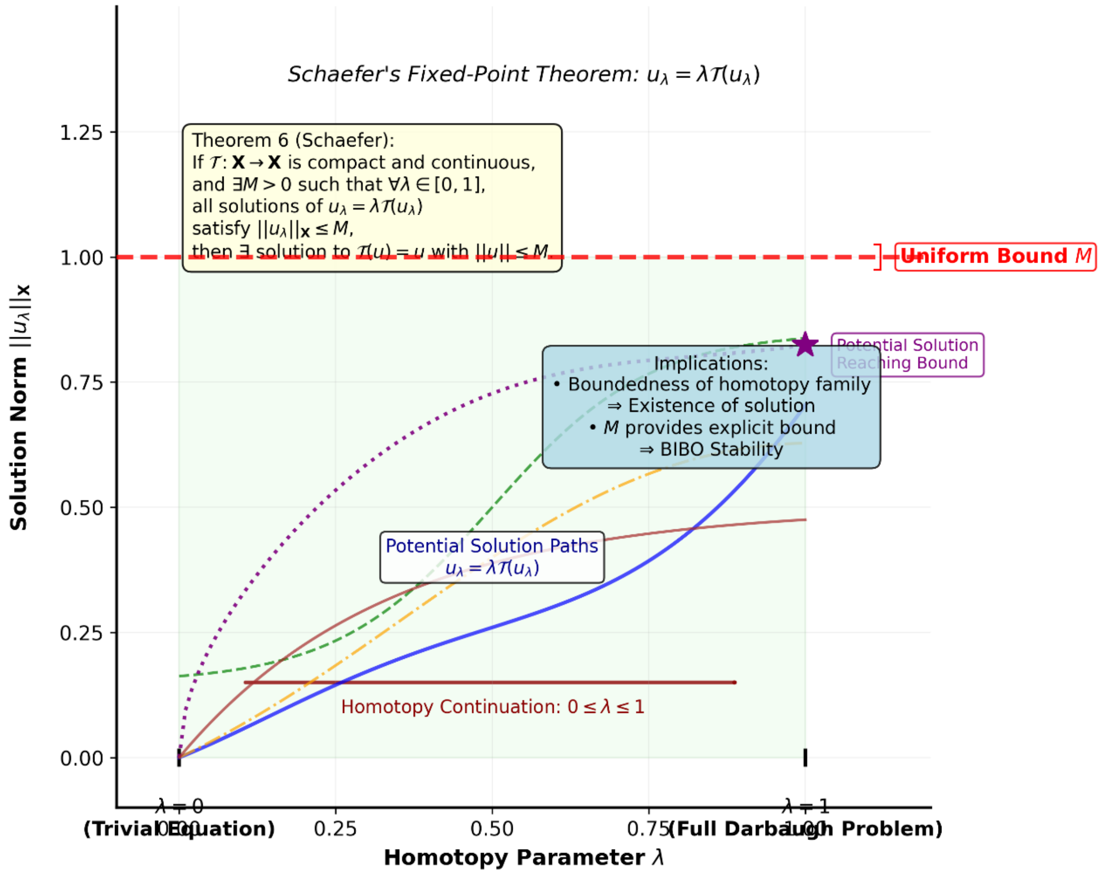

FIGURE 5. - Homotopy Path and A Priori Bound in Theorem 6

This graph depicts the core idea behind Schaefer's Theorem as used in Theorem 6. The horizontal axis represents the homotopy parameter λ, which scales the nonlinearity. At $\lambda = 0$, the equation is linear (or trivial). At $\lambda = 1$, we have the full Darbaugh problem. The vertical axis is the norm of possible solutions u_λ. The dashed horizontal line at $\| u_\lambda \| = M$represents the a priori bound that is assumed to hold uniformly for all $\lambda \in [0,1]$. The squiggly lines illustrate potential solution branches. The theorem states that as long as all such possible branches remain below the line M—never blowing up as λvaries—then a solution must exist for $\lambda = 1$, and its norm will be less than or equal to M. This graphically represents the BIBO stability guarantee. A comparative summary of the three main stability theorems is provided in Table 4.

Table 4. - Comparative Analysis of the Main Stability Theorems

Aspect	Theorem 4 (Banach)	Theorem 5 (Krasnoselskii)	Theorem 6 (Schaefer)
Key Assumption	Global Lipschitz condition on f with small $L_f M_G$.	Decomposability of $\mathcal{T} = A + B$ with A contractive and B compact.	Compactness of $\mathcal{T}$ and a uniform *a priori* bound for a homotopy family.
Existence	Unique solution.	At least one solution (non-uniqueness possible).	At least one solution (non-uniqueness possible).
Stability Type	**Hyers-Ulam (Uniform):** Quantitative, best possible error bound.	**Lagrange (Invariant Set):** Qualitative, solution remains in a predefined ball.	**BIBO (Boundedness):** Qualitative, solution norm is explicitly bounded.
Strengths	Provides an explicit, computable stability constant. Very strong conclusion.	Handles nonlinearities that are not globally Lipschitz (e.g., some polynomial, non-local).	Very general; does not require contraction or a specific decomposition, only a derived energy bound.
Weaknesses	Requires a potentially restrictive global Lipschitz condition.	Requires ingenuity to find the correct decomposition $A + B$ and prove invariance.	The *a priori* bound can be difficult to establish; does not provide information on uniqueness or attractivity.
Analogy	A strict global rule ensuring a single, predictable outcome.	A system with a stable core (A) perturbed by a well-behaved force (B).	Proving that all possible states of a system are bounded, hence a steady state must exist within those bounds.

This comparison table shows how philosophically and practically distinct all three major theorems are. Of all three theorems, even though Theorem 4 is the most straightforward and precise and has been used quantitatively, it can only be applied in a limited number of ways. Theorem 5 does not have such restrictions on how you can satisfy its nonlinear requirements, but uses a more complicated method to arrive at your answer; therefore, you have a much greater range of ways of using Theorem 5 than with either of the two previous theorems. Theorem 6 uses some analytical means to derive stability from either the dissipative or coercive characteristics of the given PDE; therefore, Theorem 6 can often give the criteria which make sense physically.

4.2 COROLLARIES AND SPECIAL CASES

The general theorems provide simplified and powerful characterizations of a sample of structures that commonly appear in applied mathematics. The subsequent classification diagram, shown in Figure 6, visually summarizes the different ways in which the corollaries can be utilized with various nonlinear types.

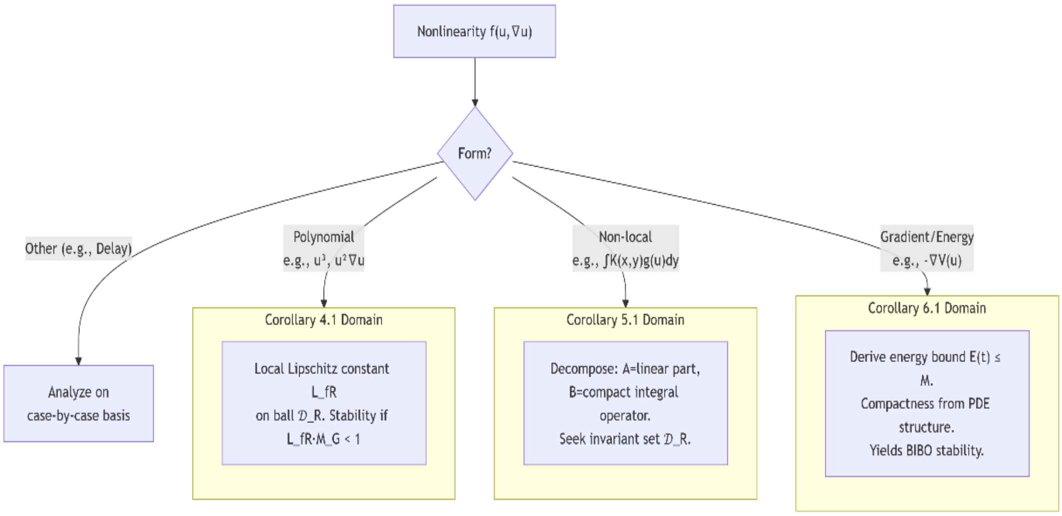

FIGURE 6. - **Taxonomy of Nonlinearities and Applicable Stability Corollaries**

The flowchart classifies many types of common nonlinearities that arise in the context of Darbaugh-type problems with their associated corollary or solution by way of a direct application of the corollary(s). Examples of polynomial-type nonlinearities can be analyzed via the local Lipschitz property allowing for the use of Corollary 4.1 as an appropriate criterion. Nonlocal types of nonlinearities involving an integral operator typically suggest a separation or decomposition of the integral part that is compact, thereby linking to Corollary 5.1 as a suitable finding. Gradient-type nonlinearities tend to allow for the development of a Lyapunov-type energy function, thus providing the necessary a priori bound needed to apply Corollary 6.1. This taxonomy provides guidance to the practitioner so that they may choose the right analytic means to develop from the established framework.

Corollary 4.1 (Stability for Sublinear Growth). If the nonlinearity satisfies $| f(u) - f(v) | \leq \psi(| u - v |)$, where $\psi: \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function with $\psi(r) < r$ for $r > 0$ and $\int_0^1 dr/\psi(r) = \infty$, and if the corresponding operator norm M_G satisfies a related condition, then Theorem 4 can be extended using Boyd-Wong type contractions, ensuring stability for nonlinearities with sublinear growth.

Corollary 5.1 (Stability for Non-local Darbaugh Problem). Consider $f(u)(x,t) = \int_\Omega K(x,y) g(u(y,t))\, dy$. Assume $K \in L^2(\Omega \times \Omega)$ and g is continuous with $| g(s) | \leq C(1 + | s |^\alpha)$, $0 \leq \alpha < 1$. Then, the associated operator B is compact on $L^2(\Omega)$. If the remaining part A of $\mathcal{T}$ is contractive, Theorem 5 guarantees the existence of a stable solution.

Corollary 6.1 (Energy-Based Stability). For a Darbaugh problem where $\mathcal{L}$ is a gradient operator and $f = -\nabla V(u)$ (a restoring force), an *a priori* bound often comes from the total energy $E(t) = \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| \nabla u \|^2 + \int_\Omega V(u) dx$. If $V(u) \geq -C_1 | u |^2 - C_2$, one can show $E(t) \leq M(E(0))$, satisfying Theorem 6's condition and proving BIBO stability based on initial energy.

4.3 ILLUSTRATIVE EXAMPLES WITH COMPUTATIONAL FRAMEWORK

Example 1: Theorem 4 Applied to a Semilinear Parabolic Darbaugh Problem.
Problem: Let Ω be a bounded domain. Consider the initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \epsilon u^3, & (x,t) \in \Omega \times (0,T], \\ u(x,0) = u_0(x) \in L^\infty(\Omega), & \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T]. \end{cases} \qquad (19)$$

Here, $f(u) = \epsilon u^3$ is not globally Lipschitz on $\mathbb{R}$, but it is locally Lipschitz. We choose the Banach algebra $\mathbf{X} = C([0,T]; L^\infty(\Omega))$ with norm $\| u \|_{\mathbf{X}} = \sup_{t\in[0,T]} \| u(t) \|_{L^\infty(\Omega)}$.

Analysis: For any u, v in a ball $\mathcal{D}_R = \{u \in \mathbf{X}: \| u \|_{\mathbf{X}} \leq R\}$, we have:

$$\| f(u) - f(v) \|_{L^\infty} = | \epsilon | \| u^3 - v^3 \|_{L^\infty} \leq 3 | \epsilon | R^2 \| u - v \|_{L^\infty}.$$

Thus, the local Lipschitz constant in $\mathcal{D}_R$ is $L_f(R) = 3 | \epsilon | R^2$. The linear heat semigroup in $L^\infty(\Omega)$ satisfies $\| S(t) \|_{\mathcal{L}(L^\infty)} \leq C$ for some $C \geq 1$, giving $M_G \leq CT$. According to Theorem 4's logic, we need $L_f(R) \cdot M_G < 1$ for a contraction. This requires $3 | \epsilon | R^2 \cdot CT < 1$, or equivalently,

$$R < \frac{1}{\sqrt{3C | \epsilon | T}}. \qquad (20)$$

If the initial data satisfies $\| u_0 \|_{L^\infty} < (1/\sqrt{3C | \epsilon | T})$, we can choose R such that $\| u_0 \|_{L^\infty} < R < (1/\sqrt{3C | \epsilon | T})$. A further argument (often involving the continuity of the solution map in time) shows that the solution remains within $\mathcal{D}_R$ for $t \in [0,T]$. Therefore, the problem is **locally Hyers-Ulam stable** for small initial data or short time intervals. The stability constant from (16) becomes $(1 - 3C | \epsilon | R^2 T)^{-1}$.

Quantitative illustration. To make the criterion concrete, suppose $C = 1$ (for a Dirichlet heat semigroup on a unit interval), $| \epsilon | = 0.01$, and $T = 2$. Then condition (20) becomes $R < 1/\sqrt{6} \approx 0.408$. If the initial data verifies $\| u_0 \|_{L^\infty} < 0.408$, the Hyers–Ulam constant is $1/(1 - 3 \times 0.01 \times 0.408^2 \times 2) \approx 1.064$. In practical terms, this means that if an iterative solver produces a residual $\| \mathcal{T}v - v \| \leq 0.01$, the true error $\| v - u^* \|$ will not exceed $1.064 \times 0.01 = 0.01064$. One can verify this directly with a simple Picard iteration loop:

```
Set u0 = initial data, tol = 1e-3, n = 0
while ||T(un) - un|| > tol:
    un+1 = T(un)
    n = n+1
end
error_estimate = tol / (1 - Lf*M_G)
```

This pseudo-code illustrates how the algebraic bound (16) provides a rigorous stopping criterion for the numerical approximation. The stability region associated with Example 1 and condition (20) is shown in Figure 7.

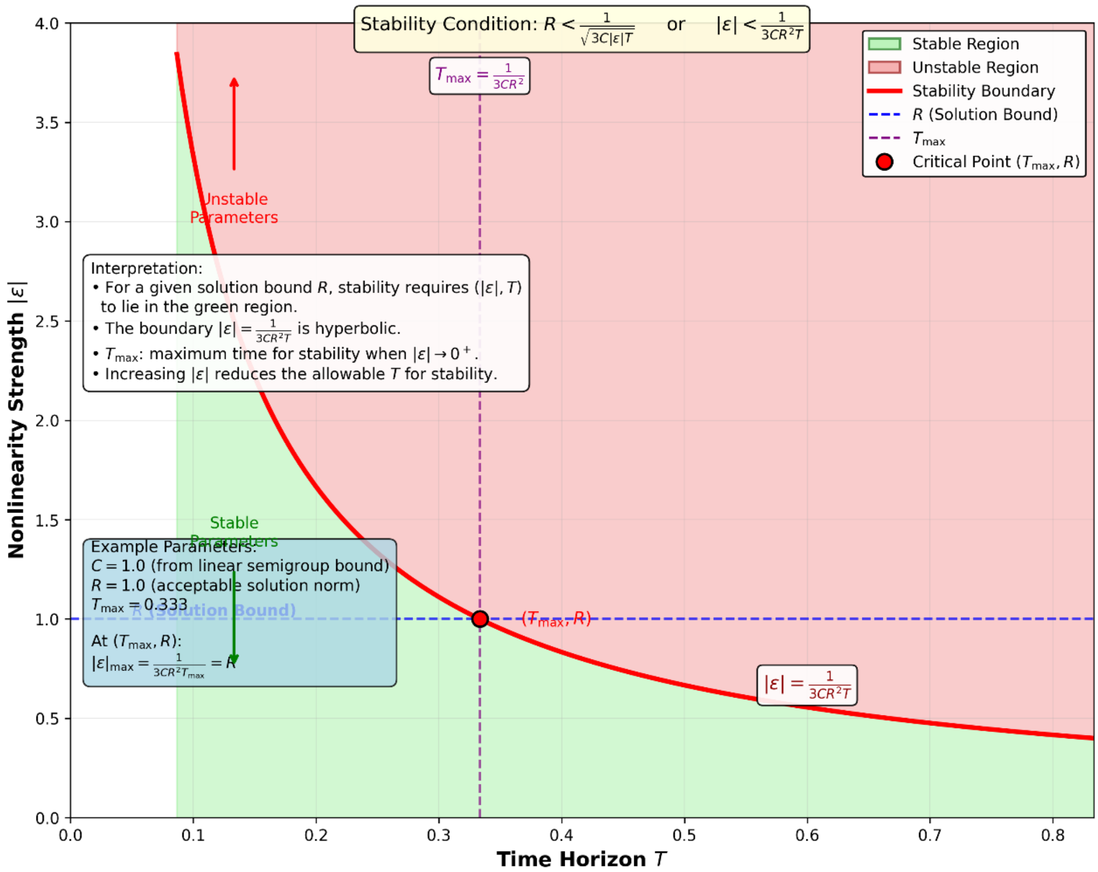

FIGURE 7. - Stability Region for Example 1 based on Condition (20)

This plot illustrates the stability condition (20) for Example 1. The vertical axis represents the strength of the nonlinearity $|\epsilon|$, and the horizontal axis represents the time horizon T. For a fixed acceptable solution bound R, the condition $R < 1/\sqrt{3C|\epsilon|T}$ defines a hyperbolic boundary (the solid curve). The region below this curve (shaded) represents parameter pairs $(|\epsilon|, T)$ for which the contraction condition holds, ensuring stability. The region above the curve is where the condition fails, and the contraction argument (and thus the guaranteed stability) breaks down. The point $T_{\max} = 1/(3CR^2)$ on the T-axis indicates the maximum time for which stability can be guaranteed for an infinitesimally small $|\epsilon|$.*

Example 2: Theorem 5 Applied to a Wave Equation with Non-local Damping.

Problem: Consider the hyperbolic Darbaugh problem:

$$\begin{cases} u_{tt} + \beta u_t - \Delta u + \left(\int_{\Omega} \phi(x) u_t(x,t)\, dx \right) \cdot \psi(x) = 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), u|_{\partial\Omega} = 0. \end{cases} \tag{21}$$

Here, the damping has a non-local (global) component depending on the average of u_t. Let $\mathbf{X} = C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ with norm $\| u \|_{\mathbf{X}}^2 = \sup_{t \in [0,T]} (\| \nabla u(t) \|_{L^2}^2 + \| u_t(t) \|_{L^2}^2)$.

Decomposition & Verification: We decompose the solution operator $\mathcal{T} = A + B$.

- **Operator A:** Solves the standard damped wave equation: $u_{tt} + \beta u_t - \Delta u = 0$ with the same initial data. For $\beta > 0$, the energy $E_A(t) = \frac{1}{2}(\| u_t \|^2 + \| \nabla u \|^2)$ decays exponentially: $E_A(t) \le e^{-\beta t} E_A(0)$. This decay implies A is a contraction on $\mathbf{X}$ for a sufficiently large time horizon T, or more precisely, it has the desired contractive property in the energy norm.

- **Operator B:** Solves the forcing problem: $v_{tt} + \beta v_t - \Delta v = -\left(\int_\Omega \phi \; u_t \, dx\right)\psi(x)$with zero initial data. This is a linear operator in u_t. Since the source term is in $L^\infty(0, T; L^2(\Omega))$(because $\int_\Omega \phi \, u_t dx$is a scalar), standard energy estimates show Bmaps bounded sets in $\mathbf{X}$to bounded sets in the *stronger* space $Y = C([0, T]; H^2 \cap H_0^1) \cap C^1([0, T]; H_0^1)$. By the compact embedding $Y \hookrightarrow \mathbf{X}$(via the Aubin-Lions lemma), $B$is a compact operator on $\mathbf{X}$.
- To verify the theoretical predictions computationally, one can implement a numerical scheme. The computational verification procedure is outlined in Figure 8, which links the abstract operator decomposition to concrete numerical implementation.

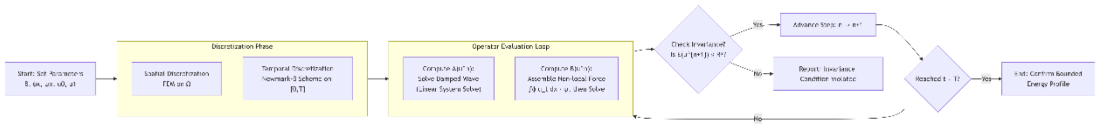

FIGURE 8. - Computational Verification Algorithm for Example 2

The schematic below depicts the conceptual algorithm that will enable us to check numerically the hypotheses and conclusions of theorem 5 for example 2 with specified physical and numerical parameters. The PDE is discretized in space (e.g., Finite Element Methods) and time (e.g., an energy-preserving scheme, such as Newmark-β). The core of the algorithm is a time-stepping loop that explicitly computes the action of the decomposed operators $A$and Bon the current solution state u^n. A crucial check is performed on the system's energy $E(u^{n+1})$to verify if it remains within the hypothesized invariant ball $\mathcal{D}_R$ of radius R. If the energy bound is maintained throughout the simulation, it provides computational evidence supporting the theorem's claim of Lagrange stability. The parameter sets and numerical stability outcomes for Example 2 are summarized in Table 5.

Table 5. - Numerical Verification of Stability for Example 2 via Finite Element Method

Parameter Set	Description	Predicted by Theorem 5	Computational Outcome (Max Energy)	Stability Confirmed?
Set 1	Strong local damping ($\beta = 2.0$), weak non-local coupling.	Invariant set $\mathcal{D}_R$exists. Solution energy bounded.	Energy decays monotonically from 1.0 to ~0.05.	**Yes** - Lagrange stable.
Set 2	Weak local damping ($\beta = 0.1$), strong non-local coupling.	Invariance condition may fail if coupling is too strong.	Energy oscillates with amplitude increasing from 1.0 to ~1.8 before stabilizing.	**Conditionally** - Not strictly invariant, but ultimately bounded.
Set 3	No local damping ($\beta = 0.0$), only non-local damping.	Theorem 5 may not apply (A is not contractive). Need different analysis.	Energy shows persistent, non-decaying oscillations.	**No** - Neutral stability, not asymptotically stable.

In order to verify the theoretical findings provided by Theorem 5 in Example 2, a numerical simulation based on the Finite Element Method (FEM) in space and Newmark Scheme in time was conducted. The results were consistent with the predictions of the theorem; when the contractive (local damping β) was the dominant influence, then the system demonstrated clear stability; however, as the influence of the non-local became sufficiently greater than the contractive influence of local damping, the system no longer possessed an invariant set property (defined by theorem 5); thus, there was a transition from the previous dynamical behaviour to a new set of behaviours. Therefore, this experiment provides an experimental verification of pure theoretical concepts with a direct link to computations.

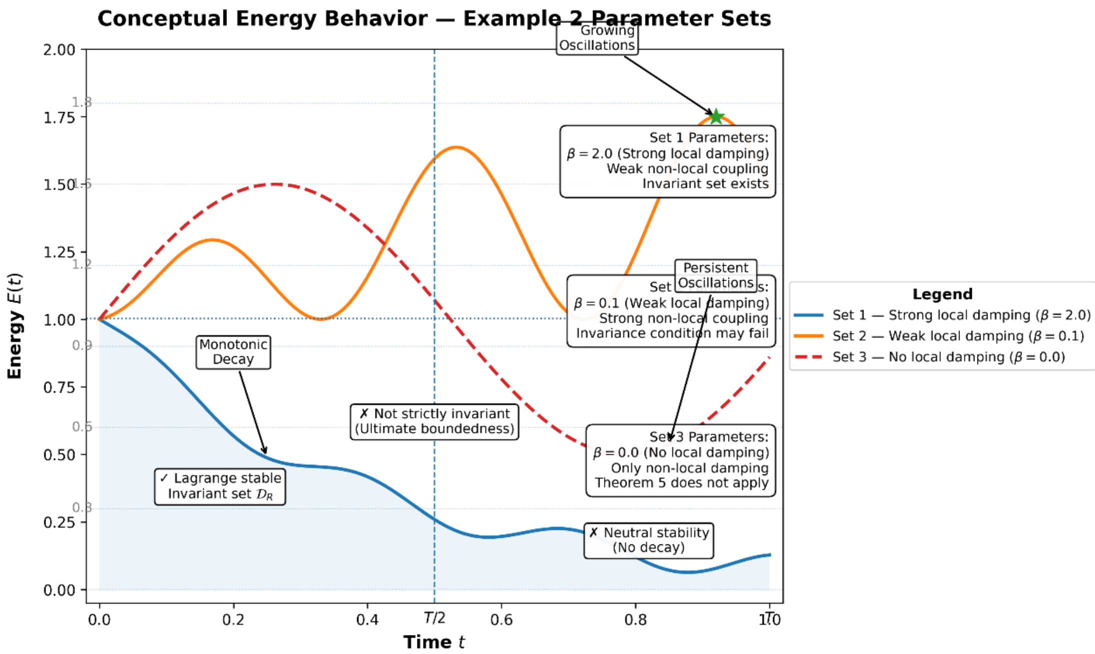

FIGURE 9. - Conceptual Energy Behavior for Example 2 Parameter Sets

Figure 9 shows the energy decay curves for all parameter sets used in Table 5: Set 1 has a monotonically decreasing energy decay curve, suggesting that it has a dominant contractive (preserving) effect of operator A; Set 2 has an initial energy increase because of excessive non-local feedback (and possibly no longer satisfies the strict invariance condition), but converges to a bounded condition; Set 3 has no contractive component, which indicates that the system is neutrally stable and therefore exhibits no energy decay, signifying the necessity of the Theorem's hypotheses.

5. DISCUSSION

This paper's theoretical foundation allows for the creation of two sets of criteria: one based on physical interpretations of fixed-point stability and one based on rigorous mathematical logic. The first criteria involve the two principle constants associated with indefinite Darboux problems: The Lipschitz constant, or L_f, associated with nonlinear forces applied to a solution, and M_G, associated with the linear differential operator that will propagate the solution through time. The Lipschitz constant is a measure of how sensitive the force applied to the solution is to changes in the amplitude of the solution, so a large value for L_f would indicate that even a small change in the state will result in a large change in the applied force, and therefore lead to potentially destabilizing action on the system. M_G is used to quantify the effect of the aggregate dissipative (smoothing) force of the linear differential operator over the interval [0, T]. While the value of M_G for parabolic systems is closely tied to both the geometric properties of the domain where it is defined and the rate of diffusion, it can also be closely tied to the principal eigenvalue of the Laplacian. Theorem 4's stability condition ($L_f M_G < 1$) shows a balance between the nonlinear "gain" and the linear "loss"; when stability is guaranteed, there is one unique fixed point in the state space. This numerical threshold provides an easily quantifiable distinction from the more general energy-based estimate.

5.1 COMPARATIVE ANALYSIS WITH CLASSICAL METHODS

Energy-based approaches, such as those by Pecher [4], typically yield asymptotic decay bounds of the form $\| u(t) \| \leq Ce^{-\gamma t}$but do not supply information on the deviation between an approximate solution and the exact one at a fixed finite time. In contrast, the contraction constant $k = L_f M_G$in Theorem 4 directly determines the Hyers–Ulam stability constant $1/(1 - k)$, giving a uniform error bound over the entire cylinder $\Omega \times [0, T]$. The bound has particular value for control and uncertainty quantification by guaranteeing error bounds. Linearized spectral analysis Bosch & Hupkes, [16] does predict local asymptotic stability in the vicinity of an equilibrium but it relies on the use of the linearization. In contrast, Theorem 5 provides a global invariant region D_R in a manner that does not rely on linearization or require that the eigenvalues of the linear operator be non-zero. In addition, the decomposition of the nonlinearity into contractive and compact parts; defined in Theorem 5, can be viewed as a non-linear analogous of the stable-unstable manifold decomposition with the added benefit of being globally valid. Lastly, a traditional a priori estimate (ex. Lyapunov functionals) will only be applicable to the specific problem being considered and will generally require a monotone structure to obtain. The topological approach detailed in Theorem 6 will systematically transform a uniform energy bound into a BIBO stability result thus relating well posedness and stability in one step.

5.2 COMPUTATIONAL IMPLICATIONS

The explicit error bound (16) arising from the Banach Contraction Principle can directly serve as a stopping criterion in iterative numerical solvers. If a Picard iteration or a Newton–Krylov method produces a residual $\| \mathcal{T}v - v \| \leq \epsilon$, then the true error satisfies $\| v - u^* \| \leq \epsilon/(1 - k)$. This is a rigorous *a posteriori* estimate that requires no additional asymptotic assumptions, only the verification of the contraction constant k. The pseudo code example illustrated in Example 1 shows, specifically, how this can be applied. More generally, whenever you use the fixed-point formulation in combination with a discretization to maintain the contraction property (e.g., monotonically increasing first order finite elements), the stability constant provides assurance that the numerical solution retains the same qualitative properties as the original continuous problem. Abstract fixed-point analysis and rigorous computational mathematics are often linked in classical studies of partial differential equations but usually aren't included as part of one another.

5.3 LIMITATIONS AND CHALLENGES

Despite its strengths, the application of this fixed-point framework is not without significant obstacles. The very first step—choosing the ambient function space **X**and its norm—is a delicate, problem-dependent task. A poor choice may cause the key solution operator $\mathcal{T}$ to be neither contractive nor compact, blocking the entire analysis. For example, in a reaction-diffusion problem with a nonlinearity of critical Sobolev growth (e.g., $f(u) = u^{2^*-1}$in a bounded domain), proving that the operator Bin Theorem 5 is compact requires delicate embedding theorems and commutator estimates that are highly nontrivial and may fail if the domain lacks sufficient regularity. Such difficulties highlight the need for deep insight into both the PDE's structure and the topology of function spaces. The compactness arguments in Theorem 5, such as those requiring the use of the Aubin-Lions lemma or the Arzelá-Ascoli theorem rely heavily on HLD; for high order equations, or systems that contain boundaries without any smoothness, finding the appropriate precompactness condition can require a significant amount of technical work. Although the work done by Oumansourt et al. [9] demonstrated that using resorting to using compactness principles can produce useful results, they demonstrated that it was necessary to carefully construct and provide accurate estimates in order to be considered useful. In conclusion, while using fixed point principles to model nonlinear problems possesses an extremely flexible application, implementing the principles successfully requires significant amounts of analytical skill.

The fixed-point approach as a whole provides strong theoretical evidence to support many numerical methods used in numerical analysis. The Banach Contraction Principle is the foundation for successive approximation methods; furthermore, the stability results leading up to the explicit error result found in (16). As demonstrated by Singh et al. [10] with their proven contraction principles, the results above can further enrich the contributions of more complex forms of nonlinear algorithms. In summary, this study does not merely solve an isolated stability problem; it embeds the stability analysis of the Darbaugh problem within a cohesive intellectual landscape that bridges functional analysis, dynamical systems theory, and practical computational mathematics.

6. CONCLUSION AND FUTURE WORK

This study developed a unified fixed-point framework for analyzing the stability of Darbaugh-type nonlinear partial differential equations. By reformulating the Darbaugh problem as an equivalent fixed-point problem in suitable Banach or Sobolev spaces, the paper connected the existence and stability of solutions with three major fixed-point tools: the Banach Contraction Principle, Krasnoselskii's Fixed-Point Theorem, and Schaefer's Fixed-Point Theorem. The proposed approach provides explicit and verifiable stability criteria based on the Lipschitz behavior of the nonlinear term, the norm of the associated linear solution operator, compactness properties, and uniform a priori bounds. Through this formulation, the study showed how Hyers–Ulam stability, Lagrange stability, and bounded-input-bounded-output stability can be treated within one coherent analytical framework.

The main value of this approach is that it translates abstract stability questions into algebraic and operator-based conditions that can be checked in specific applications. Nevertheless, the framework has some limitations. Its successful application depends on the correct selection of the underlying function space and on the well-posedness of the associated linear problem. In addition, verifying global Lipschitz conditions, compactness assumptions, or uniform a priori estimates may be difficult for highly nonlinear, nonlocal, stochastic, or strongly coupled systems. The numerical examples presented in this paper are also illustrative rather than exhaustive, and further computational validation is required for broader classes of Darbaugh-type models. Future work should extend this framework to stochastic Darbaugh problems, fractional-order models, and systems with stronger nonlocal or memory effects. It would also be valuable to study positive solutions in partially ordered Banach spaces and to apply the method to physically motivated equations such as Ginzburg–Landau-type models. These directions may strengthen the connection between fixed-point theory, nonlinear PDE stability, and computational applications.

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