Jordan-Lie Inner Ideals of the Orthogonal Simple Lie Algebras

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Abstract—Let $A$ be an associative algebra over a field $\mathbb{F}$ of any characteristic with involution $\ast$ and let $K = \text{skew}(A) = \{a \in A | a^* = -a\}$ be its corresponding sub-algebra under the Lie product $[a,b] = ab - ba$ for all $a,b \in A$. If $A = \mathcal{E}ndV$ for some finite dimensional vector space over $\mathbb{F}$ and $\ast$ is an adjoint involution with a symmetric non-alternating bilinear form on $V$, then $\ast$ is said to be orthogonal. In this paper, Jordan-Lie inner ideals of the orthogonal Lie algebras were defined, considered, studied, and classified. Some examples and results were provided. It is proved that every Jordan-Lie inner ideals of the orthogonal Lie algebras is either $B = eK e^*$ or $B$ is a type one point space.

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1 Introduction

Let $A$ be a finite dimensional associative algebra over a field $\mathbb{F}$. Recall that $A$ becomes a Lie algebra $A^{(-)}$ under the Lie bracket defined by $[x,y] = xy - yx$ for all $x,y \in A$. Suppose that $A$ has an involution $\ast$. Recall that an involution is a linear transformation $\ast$ of an algebra $A$ satisfying $(a^*)^* = a$ and $(ab)^* = b^* a^*$ for all $a,b \in A$. We denote by $K = \text{skew}(A) = \{a^* = -a | a \in A\}$ to be the vector space of the skew symmetric elements of $A$. Recall that $K$ is a Lie algebra with the Lie bracket defined by $[x,y] = xy - yx$ for all $x,y \in K$. If the characteristic of $\mathbb{F}$ is non-equal 2, then $K$ can be represented in the form:

$$K = \text{skew}(A,\ast) = \{a - a^* | a \in A\}. \quad (1.1)$$

Benkart was the first to introduce an inner ideal of a Lie algebra. She defined it as a subspace $B$ of a Lie $L$ such that the space $[B,[B,L]]$ is a subset of $B$ [4]. She highlighted the relationship between inner ideals and an $ad$-nilpotent elements [3]. Recall that an adjoint map $ad : L \rightarrow \mathfrak{gl}(L)$ is a representation from a Lie $L$ into its general linear algebra defined by $ad(\ell) = ad_{\ell}$, where $ad_{\ell} : L \rightarrow L$ is a linear transformation defined by $ad_{\ell}(x) = [\ell,x]$ for all $x \in L$. By restricting $ad$-nilpotent
elements, one can classify non-classical from classical simple Lie algebras over algebraically closed fields of characteristic ≠ 2, 3. Therefore, inner ideals play a role in classifying these algebras. Commutative inner ideals have proved to be a useful tool for classifying both finite and infinite-dimensional simple Lie algebras. It is proved in [9] that inner ideals play a role similar to one-sided ideal in associative algebras and can be used to construct Artinian structure theory for Lie algebras. Inner ideals is an essential tool in the classification of Lie algebras. (see [8] and [9]). Inner ideals of classical type Lie sub-algebras of associative(simple) rings were studied by Benkart and Fernandez Lopez (see[6]) . Baranov and Shlaka [2] in 2019 classified Jordan-Lie inner ideals of the Lie sub-algebras of finite dimensional associative algebras. An inner ideal $B$ of $A^{(k)}$ or $K^{(k)}$ is said to be Jordan-Lie if $B^2 = 0$. In recent paper, Shlaka and Mousa [11], studied Jordan-Lie inner ideals $A^{(k)}$ in the case when $A$ is simple over an algebraically closed fields of positive characteristic. Jordan-Lie inner ideals of the Lie algebras $K^{(k)}$ in the case when $A$ is simple with the symplectic involution over an algebraically closed fields of positive characteristic were also been studied by Kareem and Shlaka in [7].

In this paper, we study inner ideals of the orthogonal Lie algebras. We start with some preliminaries in section 2. Section 3 is devoted to proof some results about Jordan-Lie inner ideals of the orthogonal Lie algebras and point space.

2 Preliminaries

Throughout this paper, $\mathbb{F}$ is a field (algebraically closed), $p \geq 0$ is the characteristic of $\mathbb{F}$, $V$ is a vector space (finite dimensional over $\mathbb{F}$), $End(V)$ is the endomorphism algebra, $so(V)$ is the orthogonal Lie algebra, $A$ is an associative algebra (finite dimensional over $\mathbb{F}$) with an involution $*$, $K = skew(A, *)$ is the Lie subalgebra of $A$ defined as (1.1), $L$ is a Lie algebra (finite dimensional over $\mathbb{F}$), $M_n(\mathbb{F})$ is the matrix algebra consisting of all $n \times n$-matrices and $so_n(\mathbb{F})$ is the orthogonal Lie algebra of matrix.

Recall that an involution * of $A$ is a linear transformation of $A$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for any $a, b \in A$ [10]. Note that * does not required to be $\mathbb{F}$-linear. On the other hand, it is obvious that * maps the center $Z$ into it self. Since the restriction of * over $\mathbb{F}$ is an automorphism of order less than or equal to 2, it maps every sub-field of $Z$ into itself. Therefore $\mathbb{F}^* = \mathbb{F}$. Here we have two possibilities which are either * is $\mathbb{F}$-linear or not. Thus, we have the following definition.
2.1 **Definition** [13, 7.2] An involution is said to be of the first kind in case that ∗ is \( \mathbb{F} \)-linear, that is the restriction of ∗ relative to \( \mathbb{F} \) is the identity. Otherwise, it is called of the second kind.

2.2 **Remark** In this paper, we consider involution of the first kind only.

2.3 **Definition** Let \( B \) be a subspace of \( L \). Then \( B \) is said to be

2. [4] A commutative inner ideal if \( B \) is an inner ideal such that \( [B, B] = 0 \).
3. [2] A Jordan-Lie inner ideal (or simply, \( J \)-Lie) if \( L = \text{skew}(A) \) and \( B \) is an inner ideal such that \( B^2 = 0 \).

2.4 **Example** Consider the associative algebra \( A = \mathcal{M}_n(\mathbb{F}) \). Then \( \{e_{ij} | 1 \leq i, j \leq n\} \) form a basis of \( A \) consisting of matrix units, where \( e_{ij} \) is the \( n \times n \)-matrix with the entry 1 in the \( ij \)-th position and zero elsewhere. Thus, the Lie algebra

\[ K = \text{skew}(A) = \langle e_{ij} | 1 \leq i, j \leq n \rangle \]

has the following basis \( \{a_{ij}, b_{ij}, c_{ij} | 1 \leq i, j \leq n\} \), where

\[ a_{ij} = (e_{ij} - e_{n+i,n+j}), \quad b_{ij} = (e_{i,n+j} - e_{n+i,j}) \quad \text{and} \quad c_{ij} = (e_{n+i,j} - e_{n+j,i}). \]

Then \( B = \mathbb{F}a_{12} \) is \( J \)-Lie of \( \text{skew}(A, ∗) \). Indeed, for any \( x, y \in B \), we have

\[ x = αa_{12} = α(e_{12} - e_{n+2,n+1}), \quad y = βa_{12} = β(e_{12} - e_{n+2,n+1}). \]

Since

\[ x.y = α(e_{12} - e_{n+2,n+1}), β(e_{12} - e_{n+2,n+1}) = 0, \]

\( B^2 = 0 \). It remains to show that \( [x, [y, ℓ]] \in B \) for each \( ℓ \in K \).

Let

\[ ℓ = \sum_{i,j=1}^n \zeta_{ij}a_{ij} + \sum_{j=1}^n \eta_{ij}b_{i,n+j} + \sum_{i,j=1}^n γ_{ij}c_{ij} \]

Then

\[ x.ℓy = α(e_{12} - e_{n+2,n+1})(\sum_{i,j=1}^n \zeta_{ij}a_{ij} + \sum_{i,j=1}^n \eta_{ij}b_{ij} + \sum_{i,j=1}^n γ_{ij}c_{ij})y \]

\[ = α \sum_{j=1}^n (\zeta_{2j}e_{1j} + γ_{2j}e_{1,n+j} - γ_{j2}e_{1,n+j} + γ_{j1}e_{n+2,n+j} - γ_{1j}e_{n+2,j} + γ_{1j}e_{n+2,j})y \]

\[ = αβ(\zeta_{21}e_{12} - e_{n+1,n+1} - γ_{21}e_{1,n+1} - γ_{21}e_{n+1,n+1} - γ_{11}e_{n+2,n+1} - γ_{11}e_{n+2,n+1}) \]

\[ = αβ(\zeta_{21}e_{12} - e_{n+2,n+1}) = αβζ_{21}a_{12} \in \mathbb{F}a_{12} = B. \]

And

\[ y.ℓx = β(e_{12} - e_{n+2,n+1})(\sum_{i,j=1}^n \zeta_{ij}a_{ij} + \sum_{i,j=1}^n \eta_{ij}b_{ij} + \sum_{i,j=1}^n γ_{ij}c_{ij})x \]
\[37 = \beta \sum_{j=1}^{n} (\zeta_{2j} e_{1j} + \eta_{2j} e_{1,n+j} - \eta_{j2} e_{1,n+j} - \zeta_{j1} e_{n+2,n+j} - \gamma_{j1} e_{n+2,j} + \gamma_{j1} e_{n+2,j}) x\]

\[= \alpha \beta (\zeta_{21} e_{12} - \eta_{22} e_{1,n+1} - \eta_{22} e_{1,n+1} - \zeta_{21} e_{n+2,n+1} - \gamma_{21} e_{n+2,2} + \gamma_{21} e_{n+2,2})\]

\[= \alpha \beta \zeta_{21} (e_{12} - e_{n+2,n+1}) = \alpha \beta a_{12} \in \mathbb{𝔽} a_{12} = B.\]

Therefore, \([x, [y, \ell]] = xy \ell - x \ell y - y \ell x + \ell y x = -x \ell y - y \ell x \in B, as required.\]

### 2.5 Definition [5] A subspace \(P\) of \(L\) is said to be point space if \([P, P] = 0\) and \(\text{ad}_x^2(L) = \mathbb{𝔽} x\) for every non zero element \(x \in P\).

Example 2.6 Let \(K = \mathfrak{s}_0(\mathbb{𝔽})\). If \(n = 1\), then

\[K = \mathfrak{s}_0(\mathbb{𝔽}) = \text{span}\{\begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_3 & 0 \\ -\alpha_1 & 0 & -\alpha_3 \end{pmatrix} | \alpha_1, \alpha_2, \alpha_3 \in \mathbb{𝔽}\}\]

has a basis are

\[\{b_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, b_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, b_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\}\]

Then, we need to show that \(b_1\), is a point space. For \(x \in \mathbb{𝔽} b_1\) we have

\[x = \begin{pmatrix} 0 & \zeta & 0 \\ 0 & 0 & 0 \\ -\zeta & 0 & 0 \end{pmatrix} \text{ for some } \zeta \in \mathbb{𝔽}.\]

Let \(\ell = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_3 & 0 \\ -\alpha_1 & 0 & -\alpha_3 \end{pmatrix} \in \mathfrak{s}_0(\mathbb{𝔽}).\]

Then, \(\text{ad}_x^2(L) = [x, [x, \ell]]\]

\[= \begin{pmatrix} -\zeta \alpha_2 & \zeta \alpha_3 & 0 \\ 0 & 0 & 0 \\ 0 & -\zeta \alpha_1 & -\zeta \alpha_2 \end{pmatrix} - \begin{pmatrix} \zeta \alpha_2 & 0 & 0 \\ 0 & -\zeta \alpha_2 & 0 \\ -\zeta \alpha_3 & -\zeta \alpha_1 & 0 \end{pmatrix}\]

\[= \begin{pmatrix} 0 & \zeta & 0 \\ 0 & 0 & 0 \\ -\zeta \alpha_3 & 0 & -\zeta \alpha_2 \end{pmatrix}\]
\[
\begin{pmatrix}
0 & \zeta^2 \alpha_2 & 0 \\
0 & 0 & 0 \\
0 & -\zeta^2 \alpha_3 & 0
\end{pmatrix} - \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\zeta^2 \alpha_2 & -\zeta^2 \alpha_3 & 0
\end{pmatrix} = \begin{pmatrix}
0 & \zeta^2 \alpha_2 & 0 \\
0 & 0 & 0 \\
-\zeta^2 \alpha_2 & 0 & 0
\end{pmatrix} \in \mathbb{F}b_1
\]

Therefore, \(\mathbb{F}b_1\) and also \(\mathbb{F}b_2\) is a point space. while \(\mathbb{F}b_2\) is not point space. We will need the following lemma. For the proof see [12].

**Lemma 2.7** [12] Let \(B\) be an \(I\)-ideal of \(L\). If \(B^2 = 0\), then

1. \(b_1 \ell b_2 + b_2 \ell b_1 \in B\) for all \(b_1, b_2 \in B\) and \(\ell \in L\).
2. \(b \ell b \in B\) for all \(b \in B\) and \(\ell \in L\).

**2.6 Definition** [10] Let \(\psi: V \times V \to \mathbb{F}\) be a nondegenerate symmetric bilinear form. For each \(x \in \text{End}V\) define \(x^*\) by the following property \(\psi(x^*(v), w) = \psi(v, x(w))\) for all \(v, w \in V\). Then the map \(*: \text{End}V \to \text{End}V\) is an involution of the algebra \(\text{End}V\), called the adjoint involution with respect to \(\psi\).

**2.7 Theorem** [10, Ch.1, introduction] The map \(\psi \mapsto *\) induced one to one correspondence between equivalence classes of nondegenerate bilinear forms on \(V\) modulo multiplication by a factor in \(\mathbb{F}^*\) and involution (of first kind) on \(\text{End}V\).

**2.8 Definition** [10] Let \(*\) be an involution of \(\text{End}V\). We say that \(*\) is orthogonal if it is adjoint to a symmetric non-alternating bilinear form on \(V\).

**2.9 Definition** [3] Let \(A\) be an associative algebra with involution \(*\) over a field \(\mathbb{F}\) and let \(a \in A\). Then we define the trace of \(a\) by \(\tau(a) = a - a^*\).

**3 Jordan Lie inner ideal of the orthogonal Lie algebras**

**3.1 Theorem** Suppose that \(A\) is simple with involution and \(p \neq 2\). Let \(x \in \text{skew}(A,*)\). Then \(x = yxy^*\) for some \(y \in \text{skew}(A,*)\).

*Proof.* We have \(x^* = -x\). Since \(A\) is \(V\)-Neumann algebra, \(x = xax\) for some \(a \in A\). Put \(y = \frac{1}{2}(a - a^*) \in \text{skew}(A,*)\). Then

\[
xyx = \frac{1}{2}x(a - a^*)x = \frac{1}{2}(xxa - xa^*x) = \frac{1}{2}(x - (xa)^*) = \frac{1}{2}(x - x^*) = \frac{1}{2}(2x) = x. \]

\[\blacksquare\]
3.2 Lemma Let $eKe^* \subseteq B$ be a subspace of $K = \text{skew}(A,\cdot)$ such that $e \in BK$ and $e^* \in KB$. If $e'$ be an idempotent in $A$ such that $ee' = e'e = 0$, then $e'Be'^* \subseteq B$.

Proof. If $e'Be'^* = 0$. Then $e'Be'^* \subseteq B$. Suppose now that $e'Be'^* \neq 0$. Then $\exists a \in B$ such that $e'ae'^* \neq 0$.

As $e \in BK$, $\exists b_1 \in B$ and $k_1 \in K$ such that $e = b_1k_1$. This implies that $e' = (b_1k_1)^* = k_1'\cdot b' = k_1b_1$

We have $a \in B$ and $eae^* \in eKe^* \subseteq B$. By Lemma 2.7, $ea + ae^* = b_1k_1a + ak_1b_1 \in B$

Therefore, $e'ae'^* \in B$, as required. \[ \square \]

Recall that $A$ is simple, so $A$ can be identified with $\text{End}(V)$ for some vector space $V$. We have the following proposition.

3.3 Proposition Let $\psi: V \times V \rightarrow F$ be a non-singular form and let $\cdot$ be an adjoint involution of $A = \text{End}(V)$. Let $e,e'$ be idempotent in $A$ such that $ee' = e'e = 0$. Suppose that $eKe^* \neq 0$. Then the following hold

For each $k \in K$ such that $eke^* \neq 0$, we have

(1) $c = k + e'ke'^* \neq 0$.
(2) $e'Ke^* = 0$.
(3) $eKe^* = 0$.

Proof. (1) Let $v \in V$ such that $\psi(v, eke^*(v)) \neq 0$. Such $v$ exists because $\psi$ is non-singular. We need to show that $\psi(e^*(v), ce^*(v)) \neq 0$. Since $ee' = 0$,

$\psi(e^*(v), ce^*(v)) = \psi(v, ece^*(v))$

$= \psi(v, e(k + e'ke'^*)e^*(v))$

$= \psi(v, eke^*(v)) + \psi(v, ee'ke'^*e^*(v))$

$= \psi(v, eke^*(v)) \neq 0$.

(2) Let $w \in e'Ke^*$. Then there is $k \in K$ such that $w = e'ke^*$. For each $v \in V$ we have

$\psi(e^*(v), we^*(v)) = \psi(v, ewe^*(v)) = \psi(v, ee'ke^*e^*(v)) = \psi(v, 0) = 0$,

so $w = e'Ke^* = 0$.

(3) Let $h \in eKe'^*$. Then there is $k \in K$ such that $h = eke'^*$. For each $v \in V$ we have
\(\psi(e^*(v), he^*(v)) = \psi(v, ehe^*(v)) = \psi(v, e(eke^*)e^*(v)) = \psi(v, 0) = 0.\)

Therefore, \(h = eke^* = 0.\) ■

The idea of the following lemma comes from McCrimmon’s paper [3].

### 3.4 Lemma

Let \(A\) be an associative algebra with involution \(*\) over a field \(\mathbb{F}\). Suppose that \(L = \text{skew}(A,*)\). Then the trace \(\tau\) that defined above by \(\tau(a) = a - a^*\) has the following properties:

1. \(\tau\) is linear.
2. \(\tau(x) \in L\) for any \(x \in L\).
3. \(\tau(aba) = \tau(ba^*a)\) for any \(a, b \in A\) and \(x \in L\).
4. \(\tau(ab + \tau(b)a^* = \tau(ab) + \tau(ba^*)\) for any \(a, b \in A\).
5. \(\tau(a)x = \tau(axa) - axa - a^*xa\) for any \(a \in A\) and \(x \in L\).

**Proof.**

1. Suppose that \(a, b \in A\) and \(a \in \mathbb{F}\). Then
   \[
   \tau(aa) = a(a - a^*) = a(a - a^*) = \tau(a);
   \]
   \[
   \tau(a + b) = (a + b) - (a + b)^* = (a - a^*) + (b - b^*) = \tau(a) + \tau(b).
   \]

Thus, \(\tau\) is linear.

2. Let \(a \in A\). Then
   \[
   (\tau(a))^* = (a - a^*)^* = a^* - a = -(a - a^*) = -\tau(a).
   \]

Therefore, \(\tau(a) \in L\).

3. Let \(a \in A\) and \(x \in L\). Then we have
   \[
   \tau(x) = \tau(xa) = xa - xa^* = xa - (x^*xa^*)^* = \tau(xa).
   \]

4. Let \(a, b \in A\). Then
   \[
   \tau(ba^*) = \tau(ba^*) + \tau(ba^*) = \tau(ab - ab^* + ba^* - b^*a^*)
   \]
   \[
   = (ab - (ab)^*) + (ba^* - (ba^*)^* = \tau(ab) + \tau(ba^*)
   \]

5. For any \(a \in A\) and \(x \in L\) we have
   \[
   \tau(a)x = (a - a^*)x - a^*xa = \tau(axa) - axa - a^*axa
   \]
   \[
   = (axa - (axa)^*) - axa - a^*xa = \tau(axa) - axa - a^*xa. \]

\[\Box\]
3.5 **Lemma** Suppose that $p \neq 2, 3$ and $K = \text{skew}(\text{End}V, \ast)$. Then the following hold:

1. If $\psi(Kv, w) = 0$ for some nonzero vectors $v, w \in V$, then $w \in \mathbb{F}v$. Consequently $Kv = v^\perp$ for any nonzero vector $v \in V$.
2. If $U$ is a subspace such that $\dim U > 1$, then $KU = V$.
3. A transformation $x \in K$ satisfies $xKx^* = 0$ if and only if $\text{rank}(x) \leq 1$.

**Proof.** (1) Suppose that $v, w \in V$ be nonzero vectors such that $\psi(Kv, w) = 0$. For the contrary we assume that $w \notin Kv$. Then we could find a linear transformation $a \in A$ such that $a(w) = 0$ and $\psi(a(v), w) \neq 0$. Note that $a - a^* \in K$. Thus,

$$0 \neq \psi(a(v), w) = \psi(a(v), w) - \psi(a(v), w) - \psi(a(v), w) = 0,$$

a contradiction. Therefore $w \in Kv$. Consequently, for any nonzero vector $v$ we have $v^\perp = Kv$.

(2) Suppose that $U$ be a subspace of $V$ such that $\dim U > 1$. Then

$$KU = \sum_{w \in U} Kw = \sum_{w \in U} w^\perp = V$$

That is, any $w^\perp$ has co-dimensional 1. Thus, if $w_1^\perp = w_2^\perp$. Then $w_1 \in Kw_2$. Hence any two independent vectors $w_i^\perp$ will span all $V$.

(3) If $x^*Kx = 0$. Then

$$0 = \psi(x^*Kx(v), v) = \psi(Kx(v), x(v))$$

for all $v \in V$.

This implies $K(x(V)) \neq V$, so by (2), we get that $\dim(x(V)) \leq 1.$

3.6 **Theorem** Let $e, e^\ast, f$ be an idempotent in $A = \text{End}V$ such that $ee^\ast = e^\ast e = 0$ and $e^\ast e = 0$. Let $e^\ast f = fe^\ast = 0$ and $e^\ast f = fe^\ast = f$. If $B = eKe^\ast$ then $B$ is a $J$–Lie.

**Proof.** Let $w = eke^\ast \neq 0$, by Theorem 3.1, $w = wz'w$ for some $z' \in K$. 

put $z = e^\ast z'e$. Then

$$wzw = w(e^\ast z'e)w = ek e^\ast e''z'ek e'' = eke^\ast z'ek e'' = wz'w$$

Let $f = zw = (e^\ast z'e)(eke^\ast) = e^\ast z'ek e^\ast$. Then

$$e^\ast f = e^\ast e''z'ek e'' = 0$$

and

$$fe^\ast = e''z'ek e^\ast = (1 - e^\ast)z'ek(1 - e^\ast)e^\ast = 0$$

$$e'' f = e'' e'' z'ek e'' = e'' z'ek e'' = f$$

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Also

\[ fe'' = e''z'ek'e''e'' = f \]

By Lemma 3.5 (3), since \( \text{rank}(f) = 1 \), so \( \text{rank}(f') = 1 \). Therefore,

\[ f*Kf = (e''z'ek'e'')K(e''z'ek'e'') = e'k*e*z'ek'e''z'ek'e'' = 0 \]

and

\[ fKf' = e''z'ek'e'Ke'k'e*ze'ke'ke'ke'ke' = 0 \]

Moreover, for any \( u \in \ker(w) \) such that \( f(u) = zw(u) = 0 \),

Then \( \ker(f') = \ker(w') \), so

\[ \ker(fw') = \ker(w') \]

Recall \( f = zw \) is idempotent of rank 1. Let \( c \in \text{Im}(w') \) such that \( c \notin \ker(w') \).

Next, we claim that

\[ B \subseteq B' = eKe* + \tau(ekf), \]

for any \( d \in B \) we have

\[ d = ede* + ede'' + e'de* + e'de'' \]

\[ = ede* + ede'' - (ede'')' \]

\[ = ede* + \tau(ede'') \]

\[ = ede* + \tau(w') \]

Since \( w'(f) = w' \in eBe'' \), we have that

\[ K = eke* + \tau(efe') = ede* + \tau(ede''f) \]

As \( e''f = f \), so

\[ K = eke* + \tau(ekf) \in eKe* + \tau(ekf) \]

put \( B' = eKe* + \tau(ekf) \). Then

\[ (e + f')K(e + f')* = (e + f')K(e* + f) \]
\[ eKe^* + eKf + f^*Ke^* + f^*Kf = eKe^* + eKf - (eKf)^* = eKe^* + \tau(eKf) \]

Let \( g = e + f^* \), then
\[ g^2 = (e + e''z'ke''')(e + e''z'ke''') = e + e''z'ke''' = g \]
and \( g^*g = (e^* + f)(e + f^*) \)

\[ = (e^* + e''z'ke''')(e + e''z'ke''') = 0 \]
Now, let \( gk_1g^*, gk_2g^* \in gKg^* \) and \( \ell \in K \). Then
\[ [gk_1g^*, [gk_2g^*, \ell]] = [gk_1g^*, gk_2g^* \ell - \ell gk_2g^*] \]
\[ = gk_1g^*gk_2g^* \ell - gk_2g^* \ell gk_1g^* - gk_1g^* \ell gk_2g^* + \ell gk_2g^*gk_1g^* \]
\[ = -2gk_1g^* \ell gk_2g^* = g(-2k_1g^* \ell gk_2g^*) \in gKg^* \]
Therefore, \( B' = gKg^* \) is an \( I \)-ideal of \( K \) and \( B' \) is \( J \)-Lie of \( K \). as required. ■

3.7 Theorem Let \( e, e', f \) be an idempotent in \( \text{End}V \) such that \( ee' = e'e = 0 \). Let \( B \) be a \( J^- \)-Lie of \( K = \text{skew}(A,*) \) such that \( bKb \neq \mathbb{F}b \) for all \( b \in B \). Then the following hold
1. \( e(V) = eKe^*K(v_0^\circ) \) for all \( v_0 \in V \).
2. \( eBe^*(V) = e(V) \).

Proof. (1) Suppose that \( bKb \neq \mathbb{F}b \). We have \( B \subseteq B' = (e + f^*)K(e + f^*)^* \)
By Lemma 3.5 (1), we have \( K(v_0) = v_0^\circ \). Thus \( eKe^*K(v_0) = eKe^*(v_0^\circ) \)
Suppose that \( \dim(e^*(v_0^\circ)) \geq 1 \).
If \( \dim(e^*(v_0^\circ)) > 1 \), then by Lemma 3.5 (2),
\[ Ke^*(v_0^\circ) = V \Rightarrow eKe^*(v_0^\circ) = e(V) \]
Therefore, \( e(V) = eKe^*(v_0^\circ) \).
and if \( \dim(e^*(v_0^\circ)) = 1 \), then there exist a non-zero \( u_0 \in V \) such that
\[ e^*(v_0^\circ) = \mathbb{F}u_0. \quad (3.2) \]
Then
\[ eKe^*(v_0^\circ) = eK(u_0) = e(u_0^\circ) \]
for all \( u \in u_0^\perp \), we have \( e(u) \in (v_0^\perp)^\perp = \mathbb{F}v_0 \), because 
\[
e(u) \in e(u_0^\perp) = eK^*v_0^\perp \subseteq B(v_0^\perp) \subseteq K(v_0^\perp) = (v_0^\perp)^\perp = \mathbb{F}v_0 \]
so 
\[
u_0^\perp = e^*(V) + \mathbb{F}v_0 \]
Thus, 
\[
eK^*(u_0) = e(u_0^\perp) = e(e^*(V) + \mathbb{F}v_0) = \mathbb{F}v_0. \quad (3.3) \]

But for any non-zero \( r \in u_0^\perp \) and \( \alpha \in \mathbb{F} \), we have \( e^*(r) = \alpha u_0 \)
\[
\alpha e^*(u_0) = e^*(\alpha u_0) = e^*(e^*(r)) = e^*(r) = \alpha u_0.
\]
so \( e^*(u_0) = u_0 \). Thus, for any \( y = ey^e \in eK^* \), we can assume that \( y(u_0) = 0 \)

Let \( y(V) \subseteq u_0^\perp \), by equation (3.3), 
\[
y(V) = ey^ee^*(V) = e(ey^e(V)) = e(y(V)) \subseteq e(u_0^\perp) = \mathbb{F}v_0 \]

By Lemma 3.5 (3), if \( y \) has rank 1, then \( y^*Ky = yKy = 0 \).

By Theorem 3.1, \( 30 \neq \ell \in K \) such that \( y = \ell y \in yKy = 0 \).

Then, \( y \in eK^* \subseteq B \). Therefore \( y \in \mathbb{F}b \), but \( eK^* = bKb \)
so \( y \in bKb \). Thus, if \( bKb \neq \mathbb{F}b \), then \( eK^*(v_0) = e(V) \), as required.

(2) for any \( \ell, \ell' \in K \), we have \( e\ell e^* \in eK^* \subseteq B \).

Let \( b'' = -[e\ell e^*, [b', \ell']] \in [B, [B, K]] \subseteq B \)
\( b' \in B \) is the same \( b' \) that satisfies \( w = eb'b'' \neq 0 \). Since 
\[
b'' = -[e\ell e^*, [b', \ell']] = -[e\ell e^*, b'\ell' - \ell'b']
\]
\[
= -(e\ell e^*b'\ell' - b'\ell'e\ell e^* - e\ell e^*b' + \ell'b'e\ell e^*)
\]
\[
eb''e^* = -e(e\ell e^*b'\ell' - b'\ell'e\ell e^* - e\ell e^*b' + \ell'b'e\ell e^*)e^*
\]
\[
= -e\ell e^*b'\ell'e^* + eb'b'\ell e\ell e^*e^* + e\ell e^*b'e^* - e\ell e^*b'e^*
\]
and \( e\ell e^*b'\ell'e^* = b\ell x\ell \ell b'\ell'e^* = 0 \). As \( (bb' = 0) \)
\[
eb''e^* = e\ell e^*b'e^* = e\ell e^*(e + e')b'e^*
\]
\[
= e\ell e^*eb'e^* + e\ell e^*(e'b'e^*)
\]
By using equation\( , (e'b'e^*) = 0 \), we have \( eb''e^* = e\ell e^*(eb'e^*) \).

Since \( w = eb'e^* \), so \( eb''e^* = e\ell e^*w \) for any \( \ell, \ell' \in K \).

Let \( v \in V \). Then \( eb''e^*(v) = e\ell e^*v\ell\ell w(v) = e\ell e^*v\ell(v_0) \)

Since \( bKb \neq \mathbb{F}b \), so we must have 
\[
eBe^*(V) = eK^*(v_0)
\]
Since \( eK^*(v_0) = e(V) \), we get that 
\[
eBe^*(V) = e(V)
\]
as required. ■
Theorem Let $\mathbf{e}$, $\mathbf{f}$ be an idempotent in $\mathbf{A} = \text{EndV}$ and let $\mathbf{B}$ be a $\mathbf{J} - \text{Lie of K} = \text{skew}(\mathbf{A}, \ast)$. Suppose that $\mathbf{bKb} = \mathbb{F} \mathbf{b}$ for all $\mathbf{b} \in \mathbf{B}$. Then $\mathbf{B}$ is a type one point space.

Proof. Suppose that $\mathbf{bKb} = \mathbb{F} \mathbf{b}$, we are going to prove that $\mathbf{B}$ is a type one point space.

Recall that $\mathbf{B} \subseteq \mathbf{B}' = \mathbf{eKe}^* + \tau(\mathbf{eKf})$, so

$$
\mathbf{B}' = \mathbf{eKe}^* + \tau(\mathbf{eKf}) = b\mathbf{xKe}^* + \tau(\mathbf{eKf})
$$

(3.4)

Then $\forall y \in \mathbf{K}$, we have

$$
cyc = (\lambda \mathbf{b} + \tau(\mathbf{eKf}))y(\lambda \mathbf{b} + \tau(\mathbf{eKf}))
$$

$$
= \lambda^2 \mathbf{bby} + \lambda \mathbf{byr}(\mathbf{eKf}) + \lambda \mathbf{eKfby} + \tau(\mathbf{eKf})y\tau(\mathbf{eKf})
$$

$$
= \lambda^2 \mathbf{bby} + \lambda \mathbf{byr}(\mathbf{eKf}) + \lambda \mathbf{eKfby} + \tau(\mathbf{eKf})(\mathbf{by})^* + \tau(\mathbf{eKf})y\tau(\mathbf{eKf})
$$

By Lemma 3.4 (3),

$$
cyc = \lambda^2 \mathbf{bby} + \lambda \mathbf{byr}(\mathbf{eKf}) + \lambda \mathbf{eKfby} + \tau(\mathbf{eKf})(\mathbf{by})^* + \tau(\mathbf{eKf})y\tau(\mathbf{eKf})
$$

$$
= \lambda^2 \mathbf{bby} + \lambda \mathbf{byr}(\mathbf{eKf}) + \lambda \mathbf{eKfby} + \tau(\mathbf{eKf})(\mathbf{by})^* + \tau(\mathbf{eKf})y\tau(\mathbf{eKf})
$$

By Lemma 3.4 (3),

$$
cyc = \lambda^2 \mathbf{bby} + \lambda \mathbf{byr}(\mathbf{eKf}) + \lambda \mathbf{eKfby} + \tau(\mathbf{eKf})(\mathbf{by})^* + \tau(\mathbf{eKf})y\tau(\mathbf{eKf})
$$

Since $\mathbf{fKf} = \mathbf{fKf} = 0$.

$$
cy = \lambda^2 \mathbf{bby} + \lambda \mathbf{byr}(\mathbf{eKf}) + \lambda \mathbf{eKfby} + \tau(\mathbf{eKf})(\mathbf{by})^* + \tau(\mathbf{eKf})y\tau(\mathbf{eKf})
$$

(3.5)

we need to calculate each term. Since $\mathbf{bKb} = \mathbb{F} \mathbf{b}$, so

$$
\mathbf{bby} = \mathbf{ab}
$$

(3.6)

$$
\tau(\mathbf{byeKf}) = \tau(\mathbf{bybxKe}^*) = \tau(\mathbf{abxKe}^*) = \tau(\mathbf{eKe}^*)
$$
\[ 46 = \alpha \tau(e \ell f) \]  

(3.7)

for the third one we have

\[ e \ell f y b = b x \ell f y b \in b A b \]

Since \( \tau(a) \in L \) for any \( a \in A \), \( \tau(e \ell f y b) \in K \), then \( b x \ell f y b \in b K b \subseteq B \). 

By Lemma 3.5 (3),

\[ \tau(e \ell f y b) = \tau(b x \ell f y b) = b \tau(x \ell f y ) b = \beta b \]  

(3.8)

for some \( \beta \in \mathbb{F} \)

for the four one we have

\[ e \ell f y e \ell f = e \ell f y b x \ell f = (e \ell f y b)(x \ell f) = 0 \]

Since \( f^\ast L f = 0 \), so \( b y f^\ast \ell e^\ast x \ell f = 0 \). Then

\[ (e \ell f)y(e \ell f) = e \ell f y b x \ell f - b y f^\ast \ell e^\ast x \ell f \]

\[ = (e \ell f y b - (e \ell f y b)^\ast)x \ell f \]

\[ = \tau(e \ell f y b)x \ell f \]

\[ \beta b(x \ell f) = \beta e \ell f \]

\[ \tau(e \ell f y e \ell f) = \beta \tau(e \ell f) \]  

(3.9)

Substituting equation 3.6, 3.7, 3.8 and 3.9 in 3.5, we get that

\[ cyc = (\lambda^2 \alpha) b + (\lambda \alpha) \tau(e \ell f) + (\lambda \beta) b + \beta \tau(e \ell f) \]

\[ = (\lambda^2 \alpha + \lambda \beta) b + (\lambda \alpha + \beta) \tau(e \ell f) \]

\[ = (\lambda \alpha + \beta) c \]

Therefore, \( c K c = \mathbb{F} c \), \( B' \) is a point space

since \( B \) is a maximal point space, so \( B = B' \)

Therefore, \( B \) is a type one point space.\]  

3.9 Theorem Suppose that \( A \) is simple with the orthogonal involution \( ^* \) defined on it. If \( p \neq 2, 3 \) and \( A \) is of dimensional greater than 16, Then every \( f \) -Lie \( B \) of \( [K, K] \) is of the form \( e K e^* \) or \( B \) is a type one point space. where \( e \) is an idempotent in \( A \) such that \( e^e = 0 \).

Proof. Let \( b \in B \), Then by Theorem 3.1, \( \exists x \in K \) such that \( b = b x b \).
Let $e = bx$. Then $e' = (bx)' = x'b' = xb$, since $B$ is $J -$ Lie, $b^2 = 0$, so $e'e = xbbx = 0$. By Lemma 3.2, $bKb \subseteq B$

Suppose that $bKb \subseteq B$ is maximal with the property. Since

$$bKb = bxbKxb \subseteq bxKxb = eKe^*;$$

$$eKe^* = bxKxb \subseteq bKb,$$

We have

$$eKe^* = bKb \subseteq B \quad \text{(3.10)}$$

Next, we need to show that $B \subseteq eKe^*$

Let $e' = 1 - e$ and $e'' = (1 - e)^* = 1 - e^*$, we have

$$b = 1b1 = (e + e')b(e' + e'') = ebe^* + ebe'' + e'be^* + e'be'' \quad \text{(3.11)}$$

First, we need to show that $e'Ke'' = 0$

It remains to show that $e'Ke'' = 0$. Assume to the contrary that $e'Ke'' \neq 0$. Then $\exists c \in K$ such that $z = e'c'e'' \neq 0$. By Lemma 3.2, $e'Ke'' \subseteq B$, so $z \in B$. Let $c = b + z \in B$. In the view of Lemma 3.3(1), we have $c \neq 0$

First, we claim that $bKb \subseteq cKc$. Since $c \in B$, by Lemma 2.7, $cKc \subseteq B$. Take any $y \in K$. Then

$$ce'yc = (b + z)e^*ye(b + z) = be^*yeb + be^*yeb + ze^*yeb + ze^*yeb$$

Since $ez = e(e'c'e'') = 0$ and $ze^* = (e'c'e'')e^* = 0$,

$$ce'yc = be^*yeb = bxbxbx = bby$$

so $ce'Kec = bKb$. As $ce'Kec \subseteq cKc$, we get that

$$bKb = ce'Kec \subseteq cKc \quad \text{(3.12)}$$

Next, we need to show that $zKz \subseteq cKc$. Take any $\ell \in K$, we have

$$ce''\ell e'c = (b + z)e^*\ell e'(b + z) = be''\ell e'b + be''\ell e'z + ze''\ell e'b + ze''\ell e'z \quad \text{(3.13)}$$

By computing mutually each term, we get that
\begin{align*}
be''e'b &= b(1 - e')\ell(1 - e)b = b\ell b - b\ell eb - be''eb + be'e'eb. \\
&= b\ell b - b\ell eb + b\ell eb b = b\ell b - b\ell eb + b\ell b = 0 \quad (3.14) \\
be''e'e'z &= b(1 - e')\ell(1 - e)z = b\ell z - b\ell ez - be'\ell z + be'\ell ez \\
&= b\ell z - b\ell eb z = b\ell z - b\ell z = 0 \quad (3.15) \\
z'\ell e'b &= z(1 - e')(1 - e)b = z\ell b - z\ell eb - z'e'eb + ze'e'eb = z\ell b - z\ell b = 0 \quad (3.16) \\
z''e'e'z &= z(1 - e')(1 - e)z = z\ell z - z\ell ez - ze''\ell z + ze''\ell ez = z\ell z \quad (3.17)
\end{align*}

By substituting equation 3.14, 3.15, 3.16 and 3.17 in 3.13, we get that
c\ell e''e'c = z\ell z. \quad (3.18)

Recall that \( z = e''e'c \in K \). By Theorem 3.1, \( \exists k \in K \) such that\( z = zkz \in zKz \). By equation 3.18, we get that \( z \in zKz \subseteq cKc \). But \( z \not\in bKb \subseteq cKc \), a contradiction. Therefore,
\[ e'Ke'' = 0 \quad (3.19) \]

Therefore, \( e'be'' = 0 \). Now we have to consider to two cases depending on \( eKe'' \) whether it is zero or not.

If \( eKe'' = 0 \), then \( (e'be')^* = eb'e'' = -ebe'' \in eKe'' \)

substituting in equation (3.11), we get that
\[ b = ebe'' + ebe'' + e'be'' = ebe'' \in eKe'' \]

Therefore, \( B = eKe'' \).

Suppose now that \( eKe'' \neq 0 \). Then \( \exists k \in K \) such that \( w = eke'' \neq 0 \). Since
\[
w'Kw = (eke'')'K(eke'')
\]
\[ = e'k'e'Ke'' \subseteq e'Ke'' = 0, \]

By Lemma 3.5 (3), \( \text{rank } w \leq 1 \), so \( \text{rank } (w) = 0 \) or \( \text{rank } (w) = 1 \). Thus, \( \text{rank } (w) = 1 \) (because \( w \neq 0 \)).

Hence, \( \text{dim } w(V) \) must be one, fix any \( v_0 \in V \) such that \( w(V) = \mathbb{F}v_0 \).

Let \( v \in V \) such that
\[
w(v) = v_0. \quad (3.20)
\]
\[ V = \text{Im } (w) + \text{Ker } (w) \]
\[ 0 = \ell(e^*Ke)k + k(e^*Ke)\ell e'' \\
= e'\ell(e^*Ke)ke'' + e'k(e^*Ke)\ell e'' \\
= (e'\ell e')K(ek)e'' + (e'ke')K(e\ell e'') \\
= w''Kw + w'Kw' \]

If \( u \in \text{Ker}(w) \), then

\[ 0 = \psi(0(v), u) = \psi((w''Kw + w'Kw')(v), u) \\
= \psi(w''Kw(v), u) + \psi(w'Kw'(v), u) \\
= \psi(Kw(v), w'(u)) + \psi(Kw'(v), w(u)) \]

Since \( u \in \text{Ker}(w) \), so \( w(u) = 0 \).

\[ = \psi(Kw(v), w'(u)) \]

By Lemma 3.5 (1), \( w'(u) \in \mathbb{F}v_0 \). Now either \( w'(u) = 0 \) or \( w'(u) \neq 0 \) for all \( u \in \text{Ker}(w) \).

If \( w'(u) = 0 \) for all \( u \in \text{Ker}(w) \), then \( \text{Ker}(w) \subseteq \text{Ker}(w') \)

But \( \text{dim}(w(v)) = \text{dim}(w'(v)) = 1 \), so \( \text{Ker}(w) = \text{Ker}(w') \)

Suppose now that \( w'(u) \neq 0 \) for some \( u \in \text{Ker}(w) \), then \( \text{Im}(w') = \mathbb{F}v_0 \subseteq \text{Im}(w) \).

Since both have dimension 1, so \( \text{Im}(w') = \text{Im}(w) = \mathbb{F}v_0 \).

Then by Theorem 3.6, \( B' \) is a \( J \)-Lie.

Now, we need to show that \( B = B' \), by Theorem 3.7,

\[ e(V) = eKe^*K(v_0) \]

and

\[ eBe''(V) = e(V) \quad (3.21) \]

we claim that \( eB'e''(V) \subseteq eBe'' \)
we have \( B' = eKe^* + \tau(eKf) \)

\[ eB'e'' = (eKe^* + \tau(eKf))e'' \]
\[ eKe'^** + eKf e'^* - (eKf)'e'^* = eKf e'^* - f^*Ke'^* = eKf e'^* \]

Since \( e'^* = 0 \). Recall that \( f e'^* = f \),

\[ eB'e^* = eBf \]

Let \( e\ell f \in eKf \)

\[ e\ell f(v) = e\ell zw(v) = e\ell z(v_0) \in eKK(v_0) = eK(v_0) = e(v_0) \in e(V) \]

because \( (w(v)) = v_0 \). By equation (3.21), \( e(V) = eBe'^* \)

\[ e\ell f(v) \in eBe'^*(V) \]. Therefore \( \exists w' \in eB e'^* \) such that \( e\ell f(v) = w'(v) \).

Since \( V = Fv_0 + Ker(w) \) and \( Ker(f) = Ker(w) = Ker(w') \) fore equation (3.1), and \( w' \in eB e'^* \).

for any \( e\ell f \in eKf = eB'e'^* \), there exist \( w' \in eB e'^* \) such that \( e\ell f = w' \in eB e'^* \)

Then \( eB'e'^* \subseteq eB e'^* \) and \( B' \subseteq B \). Therefore \( B' = B \).

There exist idempotent \( (e + f^*) \) such that \( B = (e + f^*)Ke(e + f^*)' \).

Now, when \( bKb = Fb \), by Theorem 3.8, \( B' = B \) is a type one point space.

Suppose that \( \text{Im}(w') = \text{Im}(w) = Fv_0 \) for any \( w' \in eB e'^* \), we need to show that \( B \)

is a type one point space

Recall that \( wzw = w, z = e''z'e \)

Let

\[ f = wz = eb'e''z'e \]

Then

\[ fe' = eb'e''z'e(1 - e) = 0 \]

and

\[ e'f = (1 - e)eb'e''z'e = 0 \]

\[ f^2 = (eb'e''z'e)(eb'e''z'e) = eb'e''z'e = f \]

\[ ef = eeb'e''z'e = eb'e''z'e = f \]

Since \( \text{rank}(f) = 1 \), so \( \text{rank}(f^*) = 1 \)

Recall that \( f^*Kf = fKf^* = 0 \). we have \( \text{Im}(w) = \text{Im}(f) \)

for any \( w' \in eB e'^* \), we have \( \text{Im}(w) = \text{Im}(f) = \text{Im}(w') \)

we have going to prove that there exist point space \( B' = eKe' + \tau(fKe'^*) \) such

that \( B = B' \)

First, we claim that \( fw' = w' \) for any \( w' \in eB e'^* \)

Let \( u \in Ker(w') \), then \( fw'(u) = 0 \)
so \( \text{Ker}(w') \subseteq \text{Ker}(f(w')) \), therefore \( \text{Ker}(w') = \text{Ker}(f(w')) \) (co-dimension 1)

Since \( \text{Im}(w') = \text{Im}(fw') \), so \( fw' = w' \) for any \( w' \in eBe'' \).

Second, we claim that

\[
B \subseteq B' = eKe^* + \tau(fKe'')
\]

take \( K \in B \), then

\[
K = eKe^* + eKe'' + eKe'' + eKe''
\]

\[
K = eKe^* + \tau(eKe'')
\]

\[
K = eKe^* + \tau(w')
\]

Since \( fw' = w' \), so

\[
K = eKe^* + \tau(fKe'')
\]

because \( fe = f \). For all \( K \in B \), we have

\[
K = eKe^* + \tau(fKe'')
\]

\[
B \subseteq B' = eKe^* + \tau(fKe'')
\]

Now, we claim that \( B \) is a point space, that is \( bKb \neq \mathbb{F}b \). Then

\[
eKe'^*(v_0) \subseteq eKe''(v_0) \subseteq fKe''(v) = \mathbb{F}v_0
\]

but

\[
eKe''K(v_0) = e(V) \neq \mathbb{F}v_0
\]

because \( \text{rank}(V) > 1 \).

Finally, we claim that \( B' = eKe^* + \tau(fKe'') \) is point space

By using equation (3.4), and our assume that \( bKb = \mathbb{F}b \), we have

\[
B' = eKe^* + \tau(fKe'')
= bxKxb + \tau(fKe'')
= bKb + \tau(fKe'') = \mathbb{F}b + \tau(fKe'')
\]

for any \( c' \in B', \exists \ell \in K, \lambda \in \mathbb{F} \) such that

\[
c' = \lambda b + \tau(f\ell e'')
\]

for all \( y \in K \), we have

\[
c'y = (\lambda b + \tau(f\ell e''))y(\lambda b + \tau(f\ell e''))
= \lambda^2 byb + \lambda by\tau(f\ell e'')yb + \tau(f\ell e'')yb + \tau(f\ell e'')y(f\ell e'')
\]

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\[ = \lambda^2 b y b + \lambda (b y \tau(f \ell e')) + \lambda \tau(f \ell e')(b y) + \tau(f \ell e')y \tau(f \ell e') \]

By Lemma 3.4 (3),

\[ = \lambda^2 b y b + \lambda \tau(b y f \ell e'') + \lambda \tau(f \ell e''(b y)) + \tau(f \ell e'' y f \ell e'') \]

\[-f \ell e''y(f \ell e'') \ast - (f \ell e'')y f \ell e'' \]

\[ = \lambda^2 b y b + \lambda \tau(b y f \ell e'') + \lambda \tau(f \ell e'' y b) + \tau(f \ell e'' y f \ell e'') \]

\[-f \ell e'' y e' f' - e' e f' y f \ell e'' \]

Since \( f K f^* = f^* K f = 0 \),

\( c' y c' = \lambda^2 b y b + \lambda \tau(b y f \ell e'') + \lambda \tau(f \ell e'' y b) + \tau(f \ell e'' y f \ell e'') \) \hspace{1cm} (3.22)

we need to calculate each term

Since \( b K b = \mathbb{B} \), so

\[ b y b = ab \] \hspace{1cm} (3.23)

\[ \tau(b y f \ell e'') = \tau(b y e f \ell e'') = \tau(b y b x f \ell e'') = \tau(a b x f \ell e'') \]

\[ = \tau(a (e f \ell e'')) = \tau(a f \ell e'') \] \hspace{1cm} (3.24)

For the third one we have

\[ f \ell e'' y b = e f \ell e'' y b = b x f \ell e'' y b \in b A b \]

Since \( \tau(a) \in L \) for any \( a \in A \), so \( \tau(x f \ell e'' y) \in K \), then \( b \tau(x f \ell e'' y) b \in b K b \subseteq B \)

By Lemma 3.5 (3),

\[ \tau(f \ell e'' y b) = \tau(b x f \ell e'' y b) = b \tau(x f \ell e'' y) b = b b \] \hspace{1cm} (3.25)

Lastly, we have

\[ f \ell e'' y f \ell e'' = f \ell e'' y e f \ell e'' = f \ell e'' y b x f \ell e'' \]

\[ = (f \ell e'')(y b)(x f \ell e'') \]

Since \( f^* L f = 0 \), so \( b y e' \ell f^* x f \ell e'' = 0 \). Then

\[ (f \ell e'')(y e f \ell e'') = f \ell e'' y b x f \ell e'' - b y e' f^* x f \ell e'' \]
\[ = (fℓe''')yb - (fℓe''')xfℓe''') \]
\[ = \tau(fℓe'''yb)xfℓe''' = \beta b(xfℓe'''') \]
\[ \Rightarrow \tau(fℓe''')yb = \beta \tau(fℓe'''') \quad (3.26) \]

Substituting equation 3.23, 3.24, 3.25 and 3.26 in equation 3.22. We get that

\[ c'y_c' = (\lambda^2 \alpha) b + (\alpha \lambda) \tau(efℓe''') + (\lambda \beta) b + \beta \tau(efℓe''') \]
\[ = (\lambda^2 \alpha + \lambda \beta) b + (\alpha \lambda + \beta) \tau(efℓe''') \]
\[ c'y_c' = (\lambda \alpha + \beta) c' \]

Therefore, \( c'Kc' = fFc' \)

\( B' \) is point space and \( B \subseteq B' \) but \( B \) is maximal. Therefore

\[ B = B' \]

4 Conclusion

Every Jordan-Lie inner ideals of the orthogonal Lie algebras is either \( B = eKe'' \) or \( B \) is a type one point space. One can find an idempotent \( e \in A \) such that this inner ideal can be written in the form \( eke'' \). We study the relationship between these algebras and their corresponding Lie ones. Also study Jordan-Lie inner ideals of these Lie algebras. Proved that every Jordan-Lie inner ideal of the orthogonal Lie algebra of an associative algebra (finite dimensional) is generated by an idempotent \( e \in A \) with the property \( e^*e = 0 \).

5 References


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