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Generalized (Θ,Θ)- Semi-derivations of Prime Near-rings

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ABSTRACT: The concept of differential algebra has been initiated before many years ago. This research topic has inspired a lot of authors to its study with different algebraic structures such as rings or semi-rings. Their studies provided many good results in this field some of which by extending from previous works and others by introducing new notion in this direction. In this article a type of rings are called prime near-rings have been considered. In particular, we introduced the idea of generalized (Θ,Θ) -semi-derivation of prime near-rings and studied the commutativity of such types of rings by using this notion.

Keywords: Derivation, Generalized derivation, Semi-derivation, Near-ring, Commutative ring.

1. INTRODUCTION

The concept of generalized derivations has been introduced by Bresar [1]. Since that time, many studies followed this direction by studying this concept from different aspects. Some of which have been considered this concept with the notion of multiplicative derivation. For example, Koc and Golbasi [2] had investigated the semi-prime near-rings with the notion of generalized multiplicative derivation in order to present some results about it. In [3] Tiwari et. al studied the commutatively of semi-prime rings in view of multiplicative generalized derivation. They discussed several differential conditions which satisfying the commutatively of semi-prime rings. Some other works have been considered the commutativity of a given ring with the concept of generalized derivations. For instance, Ashraf and Ali [4] presented some differential conditions that gives the commutatively property of prime ring by using the notion of generalized derivation. In same year, Shuliang [5] presented some different conditions of [4] that forces the prime ring to be commutative with generalized derivation. Furthermore, some other conditions that tested the commutatively of prime and semi-prime ring have been provided by Rehman et. al [6] with the concept of generalized derivation. In addition, the commutativity and anti-commutativity of prime ring by using the same concept have been checked by Bell and Rehman [7]. On the other hand, some works had considered the notion of generalized semi-derivation to discuss the commutativity of a given ring. Bharathi and Ganesh [8] followed this direction by considering an algebraic structure namely σ -prime ring and tested its commutativity. In particular, they proved that whenever A admitting generalized semi-derivation with some differential conditions, then its commutative. Moreover, the notion of generalized left semi-derivations of prime rings has been provided by Hartinger and Mamouni [9]. The authors presented some results that concerned the commutativity of prime ring by using the mentioned notion. Some recent works on the theory of derivation have been presented in [10-13]. Finally, some conditions which satisfying the commutatively property of prime near-rings with the concept of generalized semi-derivation are given by Boua et. al [14]. Motivated by the works of the previous authors, in this paper we introduced the notion of generalized (Θ,Θ) -semi-derivation of prime near-rings which extended from [14] and studied the commutativity property in view of this notion. The present paper is organized as follows. Section two includes some basic concepts which have been used in this study. The main results of this paper are given in section three. Section four followed by the conclusions of the present article.

2. BASIC CONCEPTS

This section deals with some basic concepts that are used in this article which are given as follows.

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Definition 2.1 [14] A self-map $\hbar : A \to A$ is called semi-derivation (S-d) of A, if there exist a mapping $\lambda : A \to A$ in which $\hbar(\varepsilon\varsigma) = \hbar(\varepsilon)\lambda(\varsigma) + \varepsilon\hbar(\varsigma) = \hbar(\varepsilon)\varsigma + \lambda(\varepsilon)\hbar(\varsigma)$ and $\hbar(\lambda(\varepsilon)) = \lambda(\hbar(\varepsilon))$ for any $\varepsilon, \varsigma \in A$.

Definition 2.2 [14] A self-map $\Upsilon : A \to A$ is called generalized semi-derivation (G-S-d) of A, if there exist a (S-d) $\hbar$ of A such that $\Upsilon(\varepsilon\varsigma) = \Upsilon(\varepsilon)\varsigma + \lambda(\varepsilon)\hbar(\varsigma) = \hbar(\varepsilon)\lambda(\varsigma) + \varepsilon\Upsilon(\varsigma)$ and $\Upsilon(\lambda(\varepsilon)) = \lambda(\Upsilon(\varepsilon))$ for any $\varepsilon, \varsigma \in A$.

Definition 2.3 [15] A ring A is called prime if $\varepsilon A \varsigma = 0$ implies $\varepsilon = 0$ or $\varsigma = 0$ for any $\varepsilon, \varsigma \in A$.

Definition 2.4 [15] A ring A is called ν -torsion free if $\nu\varepsilon = 0$ implies $\varepsilon = 0$ for any $\varepsilon \in A$ and $\nu \in N$.

Definition 2.5 [15] A triple $(A, +, \cdot)$ is called left (right) near ring if it satisfied the conditions below

- $(A, +)$ is a group,
- $(A, \cdot)$ is a semi-group,
- The left (right) distributive holds.

3. MAIN RESULTS

In this section, we presented the main results of this paper. We begin with the definition of generalized (Θ, Θ) -semi-derivation which is given below.

Definition 3.1 A self-map $\Upsilon : A \to A$ is called generalized (Θ, Θ) -semi-derivation (G- (Θ, Θ) -S-d) of A, if there is a semi-derivations $\hbar$ of A associate with the function λ of A in which $\Upsilon(\varepsilon\varsigma) = \Upsilon(\varepsilon)\Theta(\varsigma) + \lambda(\varepsilon)\hbar(\varsigma) =$
$\hbar(\varepsilon)\lambda(\varsigma) + \Theta(\varepsilon)\Upsilon(\varsigma)$ and $\Upsilon(\lambda(\varepsilon)) = \lambda(\Upsilon(\varepsilon))$ for any $\varepsilon, \varsigma \in A$ with Θ is an automorphism of A.

Example 3.1 LetDefine the maps $S = \left\{ \begin{bmatrix} 0 & \varepsilon & \varsigma \\ 0 & 0 & 0 \\ 0 & 0 & \zeta \end{bmatrix} \mid \varepsilon, \varsigma, \zeta \in A \right\}$. ring and suppose that -be two torsion free left near A

$\Upsilon, \hbar, \lambda, \Theta : A \to A$ by $\Upsilon\left(\begin{bmatrix} 0 & \varepsilon & \varsigma \\ 0 & 0 & 0 \\ 0 & 0 & \zeta \end{bmatrix}\right) = \begin{bmatrix} 0 & \varepsilon\varsigma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \hbar\left(\begin{bmatrix} 0 & \varepsilon & \varsigma \\ 0 & 0 & 0 \\ 0 & 0 & \zeta \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & \varsigma \\ 0 & 0 & 0 \\ 0 & 0 & \zeta \end{bmatrix}, \lambda\left(\begin{bmatrix} 0 & \varepsilon & \varsigma \\ 0 & 0 & 0 \\ 0 & 0 & \zeta \end{bmatrix}\right) = \begin{bmatrix} 0 & \varepsilon & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and $\Theta\left(\begin{bmatrix} 0 & \varepsilon & \varsigma \\ 0 & 0 & 0 \\ 0 & 0 & \zeta \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & \varepsilon\varsigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then, A. d of -S- (Θ, Θ) -is a G Υ

Lemma 3.1 Let A be a prime near-ring and Υ be a G- (Θ, Θ) -S-d of A. Then,

$$(\Upsilon(\varepsilon)\Theta(\varsigma) + \lambda(\varepsilon)\hbar(\varsigma))\zeta = \Upsilon(\varepsilon)\Theta(\varsigma)\zeta + \lambda(\varepsilon)\hbar(\varsigma)\zeta$$

for any $\varepsilon, \varsigma, \zeta \in A$.

Proof: Let $\varepsilon, \varsigma, \zeta \in A$ then by Definition 3.1, we have

$\Upsilon((\varepsilon\varsigma)\zeta) = \Upsilon(\varepsilon\varsigma)\Theta(\zeta) + \lambda(\varepsilon\varsigma)\hbar(\zeta)$

(1) $\varepsilon, \varsigma, \zeta \in A$ for all $= \Upsilon(\varepsilon)\Theta(\varsigma)\Theta(\zeta) + \lambda(\varepsilon)\hbar(\varsigma)\Theta(\zeta) + \lambda(\varepsilon\varsigma)\hbar(\zeta)$

Also, $\Upsilon(\varepsilon(\varsigma\zeta)) = \Upsilon(\varepsilon)\Theta(\varsigma\zeta) + \lambda(\varepsilon)\hbar(\varsigma\zeta)$

(2) $\varepsilon, \varsigma, \zeta \in A$ for all $= \Upsilon(\varepsilon)\Theta(\varsigma)\Theta(\zeta) + \lambda(\varepsilon)\hbar(\varsigma\zeta)$

From (1) and (2), we obtained

$\Upsilon(\varepsilon)\Theta(\varsigma)\Theta(\zeta) + \lambda(\varepsilon)\hbar(\varsigma)\Theta(\zeta) + \lambda(\varepsilon\varsigma)\hbar(\zeta) = \Upsilon(\varepsilon)\Theta(\varsigma)\Theta(\zeta) + \lambda(\varepsilon)\hbar(\varsigma\zeta)$ for any $\varepsilon, \varsigma, \zeta \in A$ (3)

Since Θ is an onto, then

$\Upsilon(\varepsilon)\Theta(\varsigma)\zeta + \lambda(\varepsilon)\hbar(\varsigma)\zeta + \lambda(\varepsilon\varsigma)\hbar(\zeta) = \Upsilon(\varepsilon)\Theta(\varsigma)\zeta + \lambda(\varepsilon)\hbar(\varsigma\zeta)$ for any $\varepsilon,\varsigma,\zeta \in A$ (4)

By letting $\lambda(\varepsilon\varsigma)\hbar(\zeta) = 0$ and replace $\hbar(\varsigma\zeta)$ by $\hbar(\varsigma)\zeta$ in (4), we get

$(\Upsilon(\varepsilon)\Theta(\varsigma) + \lambda(\varepsilon)\hbar(\varsigma))\zeta = \Upsilon(\varepsilon)\Theta(\varsigma)\zeta + \lambda(\varepsilon)\hbar(\varsigma)\zeta$ for any $\varepsilon,\varsigma,\zeta \in A$. Therefore, as required. □

Theorem 3.1 Let A be two torsion free prime near-ring and Υ be a G-(Θ,Θ)-S-d associate with the non-zero S-d $\hbar$ of A in which $\Upsilon(\varepsilon\varsigma) = \pm\Theta(\varepsilon\varsigma)$ for each $\varepsilon,\varsigma \in A$. Then, A is commutative.

Proof: We have $\Upsilon(\varepsilon\varsigma) = \pm\Theta(\varepsilon\varsigma)$ for each $\varepsilon,\varsigma \in A$. That is mean

$\Upsilon(\varepsilon\varsigma) \pm \Theta(\varepsilon\varsigma) = 0$ for each $\varepsilon,\varsigma \in A$ (5)

Replace ς by $\varsigma\zeta$ in (5), we get

$\Upsilon(\varepsilon\varsigma\zeta) \pm \Theta(\varepsilon\varsigma\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (6)

Definition 3.1 gives that

$\Upsilon(\varepsilon\varsigma)\Theta(\zeta) + \lambda(\varepsilon\varsigma)\hbar(\zeta) \pm \Theta(\varepsilon\varsigma\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (7)

Using Lemma 3.1, we get

$(\pm\Theta(\varepsilon\varsigma) + \Upsilon(\varepsilon\varsigma))\Theta(\zeta) + \lambda(\varepsilon\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (8)

Using (5), we have

$\lambda(\varepsilon\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (9)

Replace $\lambda(\varepsilon\varsigma)$ by $\Upsilon(\varsigma\varepsilon\varsigma)$ in (9), we have

$\Upsilon(\varsigma\varepsilon\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (10)

Using Definition 3.1, we get

$\Upsilon(\varsigma\varepsilon)\Theta(\varsigma)\hbar(\zeta) + \lambda(\varsigma\varepsilon)\hbar(\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (11)

Again by Definition 3.1, (11) will be

$\Upsilon(\varsigma)\Theta(\varepsilon)\Theta(\varsigma)\hbar(\zeta) + \lambda(\varsigma)\hbar(\varepsilon)\Theta(\varsigma)\hbar(\zeta) + \lambda(\varsigma\varepsilon)\hbar(\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (12)

From the other side,

$\Upsilon(\varsigma)\Theta(\varepsilon\varsigma)\hbar(\zeta) + \lambda(\varsigma)\hbar(\varepsilon\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (13)

According to (12) and (13), we obtained

$\lambda(\varsigma)\hbar(\varepsilon)\Theta(\varsigma)\hbar(\zeta) + \lambda(\varsigma\varepsilon)\hbar(\varsigma)\hbar(\zeta) - \lambda(\varsigma)\hbar(\varepsilon\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (14)

Since λ is an onto, we get

$\varsigma(\hbar(\varepsilon)\Theta(\varsigma) - \hbar(\varepsilon\varsigma))\hbar(\zeta) + \varsigma\varepsilon\hbar(\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (15)

In (15), replace $\hbar(\varepsilon)\Theta(\varsigma) - \hbar(\varepsilon\varsigma)$ by $\varepsilon\hbar(\varsigma)$ we get

$2\varsigma\varepsilon\hbar(\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (16)

Definition 2.4 gives that

$\varsigma\varepsilon\hbar(\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (17)

From (17) and (9), we have

$\varepsilon\varsigma\hbar(\varsigma)\hbar(\zeta) = \varsigma\varepsilon\hbar(\varsigma)\hbar(\zeta)$ for each $\varepsilon,\varsigma,\zeta \in A$ (18)

That is mean $[\varepsilon,\varsigma]\hbar(\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (19)

Replace $\hbar(\varsigma)$ by $\omega\hbar(\varsigma)$ in (19), we have

$[\varepsilon,\varsigma]\omega\hbar(\varsigma)\hbar(\zeta) = 0$ for each $\varepsilon,\varsigma,\zeta \in A$ (20)

Since $\hbar \neq 0$ then Definition 2.3 forces $[\varepsilon,\varsigma] = 0$ for any $\varepsilon,\varsigma \in A$. Therefore, A is commutative. □

Corollary 3.1 Let A be two torsion free prime near-ring and Υ be a G-(Θ,Θ)-S-d associate with the non-zero S-d $\hbar$ of A in which $\Upsilon(\varepsilon\varsigma) = \pm\Theta(\varsigma\varepsilon)$ for each $\varepsilon,\varsigma \in A$. Then, A is commutative.

Theorem 3.2 Let A be prime near-ring and Υ be a G-(Θ,Θ)-S-d associate with the non-zero S-d $\hbar$ of A in which $\Upsilon(\varepsilon)\Upsilon(\varsigma) = \pm\Theta(\varepsilon\varsigma)$ for each $\varepsilon,\varsigma \in A$. Then, A is commutative.

Proof: We have $\Upsilon(\varepsilon)\Upsilon(\varsigma) = \pm\Theta(\varepsilon\varsigma)$ for each $\varepsilon,\varsigma \in A$ (21)

Replace ς by $\varsigma\zeta$ in (21), we have

$\Upsilon(\varepsilon)\Upsilon(\varsigma\zeta)=\pm\Theta(\varepsilon\varsigma\zeta)$ for each $\varepsilon,\varsigma,\zeta\in A$ (22)

Using Definition 3.1, we have

$\Upsilon(\varepsilon)\Upsilon(\varsigma)\Theta(\zeta)+\Upsilon(\varepsilon)\lambda(\varsigma)\hbar(\zeta)=\pm\Theta(\varepsilon\varsigma\zeta)$ for each $\varepsilon,\varsigma,\zeta\in A$ (23)

Using (21), we have

$\Theta(\varepsilon\varsigma\zeta)+\Upsilon(\varepsilon)\lambda(\varsigma)\hbar(\zeta)=\pm\Theta(\varepsilon\varsigma\zeta)$ for each $\varepsilon,\varsigma,\zeta\in A$ (24)

That is $\Upsilon(\varepsilon)\lambda(\varsigma)\hbar(\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (25)

Replace ε by $\varepsilon\varsigma\zeta$ in (25), and using Definition 3.1, we have

$\Upsilon(\varepsilon\varsigma)\Theta(\zeta)\lambda(\varsigma)\hbar(\zeta)+\lambda(\varepsilon\varsigma)\hbar(\zeta)\lambda(\varsigma)\hbar(\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (26)

Again, by Definition 3.1, we get

$\Upsilon(\varepsilon)\Theta(\varsigma)\Theta(\zeta)\lambda(\varsigma)\hbar(\zeta)+\lambda(\varepsilon)\hbar(\varsigma)\Theta(\zeta)\lambda(\varsigma)\hbar(\zeta)+\lambda(\varepsilon\varsigma)\hbar(\zeta)\lambda(\varsigma)\hbar(\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (27)

On the other hand,

$\Upsilon(\varepsilon)\Theta(\varsigma\zeta)\lambda(\varsigma)\hbar(\zeta)+\lambda(\varepsilon)\hbar(\varsigma\zeta)\lambda(\varsigma)\hbar(\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (28)

From (27) and (28), we have

$\lambda(\varepsilon)\hbar(\varsigma)\Theta(\zeta)\lambda(\varsigma)\hbar(\zeta)+\lambda(\varepsilon\varsigma)\hbar(\zeta)\lambda(\varsigma)\hbar(\zeta)-\lambda(\varepsilon)\hbar(\varsigma\zeta)\lambda(\varsigma)\hbar(\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (29)

Setting $\Theta(\zeta)=0$ in (29) and λ is an onto, we have

$(\varepsilon\varsigma\hbar(\zeta)-\varepsilon\hbar(\varsigma\zeta))\varsigma\hbar(\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (30)

Replace $\varepsilon\hbar(\varsigma\zeta)$ by $\varsigma\varepsilon\hbar(\zeta)$ in (30), we have

$(\varepsilon\varsigma-\varsigma\varepsilon)\hbar(\zeta)\varsigma\hbar(\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (31)

That is $[\varepsilon,\varsigma]\hbar(\zeta)\varsigma\hbar(\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (32)

Replace $\hbar(\zeta)\varsigma$ by $\omega\hbar(\zeta)$ in (32), we get

$[\varepsilon,\varsigma]\omega\hbar^2(\zeta)=0$ for each $\varepsilon,\varsigma,\zeta,\omega\in A$ (33)

Since $\hbar\neq 0$ then, Definition 2.3 forces $[\varepsilon,\varsigma]=0$ for any $\varepsilon,\varsigma\in A$. Therefore, A is commutative. □

Corollary 3.2 Let A be prime near-ring and Υ be a G-(Θ,Θ)-S-d associate with the non-zero S-d $\hbar$ of A in which $\Upsilon(\varepsilon)\Upsilon(\varsigma)=\pm\Theta(\varsigma\varepsilon)$ for each $\varepsilon,\varsigma\in A$. Then, A is commutative.

Theorem 3.3 Let A be prime near-ring and Υ be a G-(Θ,Θ)-S-d associate with the non-zero S-d $\hbar$ of A in which $\Upsilon(\varepsilon)\Upsilon(\varsigma)=\pm\Theta([\varepsilon,\varsigma])$ for each $\varepsilon,\varsigma\in A$. Then, A is commutative.

Proof: We have $\Upsilon(\varepsilon)\Upsilon(\varsigma)=\pm\Theta([\varepsilon,\varsigma])$ for each $\varepsilon,\varsigma\in A$ (34)

Replace ς by $\varsigma\varepsilon$ in (34) and using Definition 3.1, we have

$\Upsilon(\varepsilon)\Upsilon(\varsigma)\Theta(\varepsilon)+\Upsilon(\varepsilon)\lambda(\varsigma)\hbar(\varepsilon)=\pm\Theta([\varepsilon,\varsigma])\Theta(\varepsilon)$ for each $\varepsilon,\varsigma\in A$ (35)

Using (34), we have

$\Upsilon(\varepsilon)\lambda(\varsigma)\hbar(\varepsilon)=0$ for each $\varepsilon,\varsigma\in A$ (36)

Replace ε by $\varepsilon\varsigma\zeta$ in (36) and using Definition 3.1, we have

$\Upsilon(\varepsilon\varsigma)\Theta(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)+\lambda(\varepsilon\varsigma)\hbar(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (37)

Again by Definition 3.1, we have

$\Upsilon(\varepsilon)\Theta(\varsigma)\Theta(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)+\lambda(\varepsilon)\hbar(\zeta)\Theta(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)+\lambda(\varepsilon\varsigma)\hbar(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (38)

From the other side,

$\Upsilon(\varepsilon)\Theta(\varsigma\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)+\lambda(\varepsilon)\hbar(\varsigma\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (39)

From (38) and (39), we obtained

$\lambda(\varepsilon)\hbar(\zeta)\Theta(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)+\lambda(\varepsilon\varsigma)\hbar(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)-\lambda(\varepsilon)\hbar(\varsigma\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in A$ (40)

By letting $\Theta(\zeta)=0$ in (40), we get

$\lambda(\varepsilon\varsigma)\hbar(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)-\lambda(\varepsilon)\hbar(\varsigma\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (41)

Since λ is an onto, then we have

$(\varepsilon\varsigma\hbar(\zeta)-\varepsilon\hbar(\varsigma\zeta))\varsigma\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (42)

Replace $\varepsilon\hbar(\varsigma\zeta)$ by $\varsigma\varepsilon\hbar(\zeta)$ in (42), we have

$(\varepsilon\varsigma-\varsigma\varepsilon)\hbar(\zeta)\varsigma\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (43)

That is $[\varepsilon,\varsigma]\hbar(\zeta)\varsigma\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (44)

Replace $\hbar(\zeta)\varsigma$ by $\omega\hbar(\zeta)$ in (44), we get

$[\varepsilon,\varsigma]\omega\hbar(\zeta)\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (45)

Since $\hbar\neq 0$ then, Definition 2.3 forces $[\varepsilon,\varsigma]=0$ for any $\varepsilon,\varsigma\in \mathsf{A}$. Thus, A is commutative. □

Theorem 3.4 Let A be prime near-ring and Υ be a G-(Θ,Θ)-S-d associate with the non-zero S-d $\hbar$ of A in which $\Upsilon(\varepsilon)\Upsilon(\varsigma)=\pm\Theta(\varepsilon\circ\varsigma)$ for each $\varepsilon,\varsigma\in \mathsf{A}$. Then, A is commutative.

Proof: We have $\Upsilon(\varepsilon)\Upsilon(\varsigma)=\pm\Theta(\varepsilon\circ\varsigma)$ for each $\varepsilon,\varsigma\in \mathsf{A}$ (46)

Replace ς by $\varsigma\varepsilon$ in (46) and using Definition 3.1, we have

$\Upsilon(\varepsilon)\Upsilon(\varsigma)\Theta(\varepsilon)+\Upsilon(\varepsilon)\lambda(\varsigma)\hbar(\varepsilon)=\pm\Theta(\varepsilon\circ\varsigma)\Theta(\varepsilon)$ for each $\varepsilon,\varsigma\in \mathsf{A}$ (47)

By using (46), we get

$\Upsilon(\varepsilon)\lambda(\varsigma)\hbar(\varepsilon)=0$ for each $\varepsilon,\varsigma\in \mathsf{A}$ (48)

Replace ε by $\varepsilon\varsigma\zeta$ in (48) and using Definition 3.1, we have

$\Upsilon(\varepsilon\varsigma)\Theta(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)+\lambda(\varepsilon\varsigma)\hbar(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (49)

Again by Definition 3.1, we have

$\Upsilon(\varepsilon)\Theta(\varsigma)\Theta(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)+\lambda(\varepsilon)\hbar(\varsigma)\Theta(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)+\lambda(\varepsilon\varsigma)\hbar(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (50)

From the other side,

$\Upsilon(\varepsilon)\Theta(\varsigma\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)+\lambda(\varepsilon)\hbar(\varsigma\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (51)

From (50) and (51), we obtained

$\lambda(\varepsilon)\hbar(\varsigma)\Theta(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)+\lambda(\varepsilon\varsigma)\hbar(\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)-\lambda(\varepsilon)\hbar(\varsigma\zeta)\lambda(\varsigma)\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (52)

Let $\Theta(\zeta)=0$ in (52) and λ is an onto, we have

$\varepsilon\varsigma\hbar(\zeta)\varsigma\hbar(\varepsilon\varsigma\zeta)-\varepsilon\hbar(\varsigma\zeta)\varsigma\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (53)

Replace $\varepsilon\hbar(\varsigma\zeta)$ by $\varsigma\varepsilon\hbar(\zeta)$ in (53), we get

$(\varepsilon\varsigma-\varsigma\varepsilon)\hbar(\zeta)\varsigma\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (54)

Replace $\hbar(\zeta)\varsigma$ by $\omega\hbar(\zeta)$ in (54), we have

$[\varepsilon,\varsigma]\omega\hbar(\zeta)\hbar(\varepsilon\varsigma\zeta)=0$ for each $\varepsilon,\varsigma,\zeta,\omega\in \mathsf{A}$ (55)

Since $\hbar\neq 0$ then, Definition 2.3 gives $[\varepsilon,\varsigma]=0$ for any $\varepsilon,\varsigma\in \mathsf{A}$. Therefore, A is commutative. □

Theorem 3.5 Let A be prime near-ring and Υ_1,Υ_2 be two G-(Θ,Θ)-S-ds associate with the non-zero S-d $\hbar$ of A in which $\Upsilon_1(\varepsilon)\Theta(\varsigma)=\Theta(\varepsilon)\Upsilon_2(\varsigma)$ for each $\varepsilon,\varsigma\in \mathsf{A}$. Then, A is commutative.

Proof: We have $\Upsilon_1(\varepsilon)\Theta(\varsigma)=\Theta(\varepsilon)\Upsilon_2(\varsigma)$ for each $\varepsilon,\varsigma\in \mathsf{A}$ (56)

Replace ς by $\varsigma\zeta$ in (56) and using Definition 3.1, we have

$\Upsilon_1(\varepsilon)\Theta(\varsigma)\Theta(\zeta)=\Theta(\varepsilon)\Upsilon_2(\varsigma)\Theta(\zeta)+\Theta(\varepsilon)\lambda(\varsigma)\hbar(\zeta)$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (57)

Using (56), we get

$\Theta(\varepsilon)\lambda(\varsigma)\hbar(\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (58)

Replace $\lambda(\varsigma)$ by $\Upsilon_1(\varepsilon\zeta\varsigma)$ in (58) and using Definition 3.1,we have

$\Theta(\varepsilon)\Upsilon_1(\varepsilon\zeta)\Theta(\varsigma)\hbar(\zeta)+\Theta(\varepsilon)\lambda(\varepsilon\zeta)\hbar(\varsigma)\hbar(\zeta)=0$ for each $\varepsilon,\varsigma,\zeta\in \mathsf{A}$ (59)

Again by Definition 3.1, we get

$$\Theta(\varepsilon)\Upsilon_1(\varepsilon)\Theta(\zeta)\Theta(\varsigma)\hbar(\zeta)+\Theta(\varepsilon)\lambda(\varepsilon)\hbar(\zeta)\Theta(\varsigma)\hbar(\zeta)+\Theta(\varepsilon)\lambda(\varepsilon\zeta)\hbar(\varsigma)\hbar(\zeta)=0 \text{ for each } \varepsilon,\varsigma,\zeta\in A \tag{60}$$

On the other hand,

$$\Theta(\varepsilon)\Upsilon_1(\varepsilon)\Theta(\zeta\varsigma)\hbar(\zeta)+\Theta(\varepsilon)\lambda(\varepsilon)\hbar(\zeta\varsigma)\hbar(\zeta)=0 \text{ for each } \varepsilon,\varsigma,\zeta\in A \tag{61}$$

From (60) and (61), we have

$$\Theta(\varepsilon)\lambda(\varepsilon)\hbar(\zeta)\Theta(\varsigma)\hbar(\zeta)+\Theta(\varepsilon)\lambda(\varepsilon\zeta)\hbar(\varsigma)\hbar(\zeta)-\Theta(\varepsilon)\lambda(\varepsilon)\hbar(\zeta\varsigma)\hbar(\zeta)=0 \text{ for each } \varepsilon,\varsigma,\zeta\in A \tag{62}$$

Putting $\Theta(\varsigma)=0$ in (62) and since λ is an onto, we have

$$\Theta(\varepsilon)\varepsilon\zeta\hbar(\varsigma)\hbar(\zeta)-\Theta(\varepsilon)\varepsilon\hbar(\zeta\varsigma)\hbar(\zeta)=0 \text{ for each } \varepsilon,\varsigma,\zeta\in A \tag{63}$$

Replace $\varepsilon\hbar(\zeta\varsigma)$ by $\zeta\varepsilon\hbar(\varsigma)$ in (63), we get

$$\Theta(\varepsilon)[\varepsilon,\zeta]\hbar(\varsigma)\hbar(\zeta)=0 \text{ for each } \varepsilon,\varsigma,\zeta\in A \tag{64}$$

Replace $\Theta(\varepsilon)[\varepsilon,\zeta]$ by $[\varepsilon,\zeta]\Theta(\varepsilon)$ and let $\Theta(\varepsilon)=\omega$ in (64), we have

$$[\varepsilon,\zeta]\omega\hbar(\varsigma)\hbar(\zeta)=0 \text{ for each } \varepsilon,\varsigma,\zeta,\omega\in A \tag{65}$$

Since $\hbar\neq 0$ then, Definition 2.3 gives $[\varepsilon,\zeta]=0$ for any $\varepsilon,\zeta\in A$. Thus, A is commutative. □

Corollary 3.3 Let A be prime near-ring and Υ be a G-(Θ,Θ)-S-d associate with the non-zero S-d $\hbar$ of A in which $\Upsilon(\varepsilon)\Theta(\varsigma)=\Theta(\varepsilon)\Upsilon(\varsigma)$ for each $\varepsilon,\varsigma\in A$. Then, A is commutative.

4. CONCLUSIONS

As a conclusion, the notion of generalized (Θ,Θ)-S-d of prime near-rings has been provided. The obtained results showed that the prime near-ring is commutative in view of the presented idea whenever the mentioned differential conditions are existed. Moreover, this study can be extended by considering some other different conditions that satisfying the commutativity property of such types of rings.

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