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Combining Laplace transform and Variational Iteration Method for Solving Singular IVPs and BVPs of Lane–Emden Type Equation

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ABSTRACT: This study uses the combined Laplace transform and variational iteration method to analytically solve initial value problems (IVPs) and boundary value problems (BVPs) of Lane- Emden- equation with singular behaviour at t=0. Initially, calculating the lagrange multiplayer in this method is required and it was calculated by integration of variation based on the extremum condition and then integrating by parts, this method is applying on several problems and the results were improved using Padé approximations of order [M/N] on the last iteration that we have it, to show the high of accuracy of the LTVIM we compared the results of the exact solution with the proposed method and Padé approximations by calculating the absolute error value between them.

Keywords: Lane-Emden equation; Initial value problems (IVPs), Boundary value problems (BVPs), Laplace Transform method; Variational iteration method, Laplace Transform Variation Iteration Method (LTVIM), Padé approximations

1. INTRODUCTION

Numerous issues found in mathematical physics literature can be uniquely expressed as Lane-Emden type equations, which are stated in the following way:

$$y'' + \frac{2}{t}y' + f(t)g(y) = h(t), y(0) = A, y'(0) = 0, 0 < t \leq 1 \quad (1)$$

where A is a constant, $f(t), h(t)$ *and* $g(y)$ are some given functions of t and y,respectively.

Lane [1,2] introduced this equation, and Emden [3] investigated it further. Stellar study of structure has been a major challenge since the inception of stellar astrophysics. Determining the material, pressure, and density rotational patterns of stars has been an ongoing endeavor. one significant outcome of these investigations is the Lane-Emden equation, which characterizes the density profile of the gaseous star. In mathematics, the second-order singular ordinary differential equation is known as the Lane-Emden equation. In astrophysics, the Lane-Emden equation can be viewed as a Poisson equation for the gravitational potential of a self-gravitating spherically symmetric polytropic fluid.

the theory of star structure, isothermal gas spheres, the thermal behavior of a spherical cloud of gas, and thermionic currents are only a few of the phenomena that have been modeled by the Lane-Emden equation in mathematical physics, thermodynamics, fluid mechanics, and astrophysics [4-8]. The initial version of the equations in the Lane-Emden type was published by Lane [9]. Emden investigated them in further detail in 1870 [10], taking into account the thermal behavior of a spherical cloud of gas that is subject to the classical principles of thermodynamics and behaves under the mutual attraction of its molecules. It is kindly asked that the reader go through [4-15] to learn more about the history, modifications, and uses of Lane-Emden-type equations.

the Lane-Emden equation was solved using various methods in numerous investigations. Wazwaz, 2001 [16] applied a new algorithm for solving differential equations of Lane-Emden type. Homotopy perturbation method (HPM) was utilized by Yildirim and Ozis, 2007 [17] to solve Lane-Emden singular IVPs. Ramos, 2008 [18] used series approach to the Lane-Emden equation and comparison with the HPM. the singular Lane-Emden type equation was analytically solved using the optimal homotopy asymptotic method (OHAM) by Iqbal and Javed (2011) [19]. Abu Arqub et al, 2013 [20] applied new analytical method to representation of the exact solution of generalized Lane-Emden equations. šmarda and khan 2015, [21] used an effective computational method for resolving singular initial value issues for equations of the Lane-Emden type. Hosseini and Abbasbandy 2015, [22] combined the spectral method and Adomian decomposition method (ADM) to solve Lane-Emden equations. Tripathi and Mishra, 2016 [23] used homotopy perturbation method with Laplace Transform (LT-HPM) for solving Lane-Emden type differential equations. Al-Hayani et al., 2017 [24] used the homotopy analysis method (HAM) in conjunction with the genetic algorithm. Parand and Hashemi, 2018 [25] used RBF-DQ method for solving non-linear differential equations of Lane-Emden. Izadi, 2021 [26] used a discontinuous finite element approximation to singular Lane-Emden –type equations. Saadeh et al.,2022[27] used a new technique Laplace transform and residual error function to building a series solution of Lane-Emden equation. AWONUSIKA and OLATUNJI, 2022 [28] used a numerical and Analytical solutions of a class of generalised Lane-Emden equations. Parand et al., 2023 [29] solved nonlinear differential equations of Lane–Emden type by neural network approach. AWONUSIKA 2024 [30] used Adomian decomposition method to get an analytical solution of a class of Lane–Emden equations.

The Laplace transformation method with variational iteration method (LTVIM) will be used in this paper. Achieving Approximate-exact solutions for several models of second order Lane-Emden equation with singular behavior at $t = 0$ is the main objective of this effort. The LTVIM handles both linear and nonlinear factors in an understandable manner without the need for any further steps. It has been shown that this method yields quickly convergent series solutions for both linear and nonlinear problems.

This paper is organized as follows: section 2, demonstrates the derivation of the proposed method, LTVIM has applied to solve problems and obtain all numerical results in section 3, and in section 4 is specific to conclusions.

2. APPLICATIONS OF LTVIM TO LANE-EMDEN EQUATION

Multiplying t and then taking the Laplace transform on both sides of Eq. (1) gives

$$\mathcal{L}[ty''(t)] + \mathcal{L}[2y'(t)] + \mathcal{L}[tf(t)g(y)] - \mathcal{L}[th(t)] = 0,$$

$$-s^2\mathcal{L}'[y(t)] - y(0) + \mathcal{L}[tf(t)g(y)] - \mathcal{L}[th(t)] = 0, \tag{2}$$

where $\mathcal{L}'[y(t)] = \frac{d}{ds}\mathcal{L}[y(t)]$, $\mathcal{L}$ is the operator of Laplace transform.

To solve Eq. (2), the VIM admits the use of the correction functional given by

$$\mathcal{L}[y_{n+1}(t)] = \mathcal{L}[y_n(t)] + \int \lambda(s;t)\{-s^2\mathcal{L}'[y_n(t)] - y_n(0) + \mathcal{L}[tf(t)g(y_n)] - \mathcal{L}[th(t)]\}ds. \tag{3}$$

Taking the variation with respect to $y_n(t)$ on both sides of the Eq. (3) leads to

$$\frac{\delta}{\delta y_n}\mathcal{L}[y_{n+1}(t)] = \frac{\delta}{\delta y_n}\mathcal{L}[y_n(t)] + \frac{\delta}{\delta y_n}\int \lambda(s;t)\{-s^2\mathcal{L}'[y_n(t)] - y_n(0) + \mathcal{L}[tf(t)g(y_n)] - \mathcal{L}[th(t)]\}ds. \tag{4}$$

Calculus of variations and integration by parts of Eq. (4) gives

$$\mathcal{L}[\delta y_{n+1}(t)] = \mathcal{L}[\delta y_n(t)][1 - t^2\lambda(t;t)] + \int \mathcal{L}[\delta y_n(t)][s^2\lambda'(s;t) + 2s\lambda(s;t)]ds. \tag{5}$$

Note that $\delta y_{n+1}(0) = 0$, we obtain the stationary conditions

$$\begin{cases} \mathcal{L}[\delta y_n]: & s^2\lambda'(s;t) + 2s\lambda(s;t) = 0, \\ \mathcal{L}[\delta y_n]: & 1 - s^2\lambda(s;t)|_{s=t} = 0, \end{cases} \tag{6}$$

for which the Lagrange multiplier λ should satisfy. Solving the system (6) for λ yields

$$\lambda(s;t) = \frac{1}{s^2}, \tag{7}$$

Substituting Eq. (7) in Eq. (3) we get

$$\mathcal{L}[y_{n+1}(t)] = \mathcal{L}[y_n(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_n(t)] - y_n(0) + \mathcal{L}[tf(t)g(y_n)] - \mathcal{L}[th(t)]\}ds, \tag{8}$$

taking inverse Laplace of Eq. (8), we obtain iterations formula

$$y_{n+1}(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_n(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_n(t)] - y_n(0) + \mathcal{L}[tf(t)g(y_n)] - \mathcal{L}[th(t)]\}ds\right\}, \qquad n \geq 0 \tag{9}$$

3. APPLICATIONS AND NUMERICAL RESULTS

In this section, we examine distinct models with singular behavior at $t = 0$, three models nonlinear IVPs and two models linear BVPs. To show the high accuracy of the approximate solution results (LTVIM) and the Padé approximation **(PA)** of order $[N/M]$ compared with the exact solution, the absolute errors between them are defined as follows:

$AE_1 = |\text{Exact Solution} - \text{LTVIM}|$,

$AE_2 = |\text{Exact Solution} - \text{Padé Approximation}|$.

With a precision of 20 digits, the computations related to the examples were carried out using the Maple 18 package.

3.1 LANE–EMDEN NON-LINEAR IVPS

Problem 1. Solve the following non-linear nonhomogeneous Lane-Emden equation by using LTVIM

$$y'' + \frac{2}{t}y' + y^3 = 6 + t^6, \quad y(0) = 0, \quad y'(0) = 0. \qquad (10)$$

Multiplying t and then taking the Laplace transform on both sides of Eq. (10) gives

$$-s^2\mathcal{L}'[y(t)] - y(0) + \mathcal{L}[ty^3(t)] - \mathcal{L}[6t + t^7] = 0, \qquad (11)$$

Operating with Laplace transform on both sides of Eq. (11) and applying by the same way proceeding as the Eqs. (3)-(9), we obtain the following recursive way

$$y_{n+1}(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_n(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_n(t)] - y_n(0) + \mathcal{L}[ty_n^3(t)] - \mathcal{L}[6t + t^7]\}ds\right\}, \qquad n \geq 0 \qquad (12)$$

Let $y_0(t) = 0$, then, from (12), we have

$$y_1(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_0(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_0(t)] - y_0(0) + \mathcal{L}[ty_0^3(t)] - \mathcal{L}[6t + t^7]\}ds\right\},$$

$$= t^2 + \frac{1}{72}t^8,$$

In the same way the other iterations

$$y_2(t) = t^2 - \frac{1}{5040}t^{14} - \frac{1}{725760}t^{20} - \frac{1}{262020096}t^{26},$$

$$\begin{aligned} y_3(t) = t^2 &+ 0.1417 \times 10^{-5}t^{20} + 0.5888 \times 10^{-8}t^{26} - 0.10099 \times 10^{-9}t^{32} \\ &- 0.1106 \times 10^{-11}t^{38} - 0.122624 \times 10^{-14}t^{44} + 0.5144 \times 10^{-16}t^{50} \\ &+ 0.4815 \times 10^{-18}t^{56} + 0.2272 \times 10^{-20}t^{62} + 0.6480 \times 10^{-23}t^{68} \\ &+ 0.1084 \times 10^{-25}t^{74} + 0.8578 \times 10^{-29}t^{80}. \end{aligned}$$

This series has the closed form as $n \to \infty$ gives t^2, i.e.,

$$y_{Exact}(t) = t^2,$$

which is the exact solution of the problem 1.

In **Table 1** present the numerical results applying the LTVIM $(\boldsymbol{y_3(t)})$, the Padé approximation (PA) of order [1/1] with the exact solution $(y_{Exact}(t))$.

$$[1/1] = \frac{a_0 + a_1 t}{1 + b_1 t} = t^2.$$

Table 1. Numerical results for problem 1

t	$y_{Exact}(t)$	$y_3(t)$	AE_1	PA [1/1]	AE_2
0.0	0.0	0.000000000	0.0	0.0	0.0
0.1	0.01	0.010000000	1.417E-26	0.01	0.0
0.2	0.04	0.040000000	1.486E-20	0.04	0.0
0.3	0.09	0.090000000	4.941E-17	0.09	0.0
0.4	0.16	0.160000000	1.558E-14	0.16	0.0
0.5	0.25	0.250000000	1.351E-12	0.25	0.0
0.6	0.36	0.360000000	5.182E-11	0.36	0.0
0.7	0.49	0.490000001	1.131E-09	0.49	0.0
0.8	0.64	0.640000016	1.635E-08	0.64	0.0
0.9	0.81	0.810000172	1.726E-07	0.81	0.0
1.0	1.00	1.000001423	1.423E-06	1.00	0.0

In Figure 1, a very good agreement is shown between the exact solution $(y_{Exact}(t))$ with a continuous line and the LTVIM $(\boldsymbol{y_3(t)})$ with the symbol o.

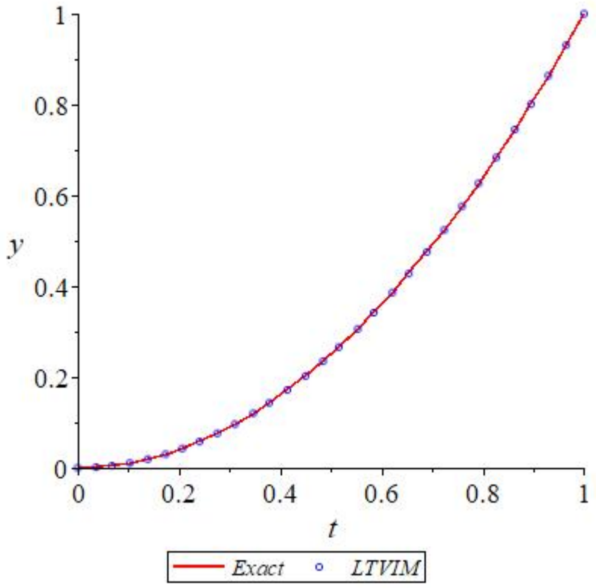

Fig. 1. Graph of $y_{Exact}(t)$ and LTVIM $(\boldsymbol{y_3(t)})$

Problem 2. Solve the following non-linear homogeneous Lane-Emden equation by using LTVIM

$$y''(t) + \frac{2}{t}y'(t) + y^5 = 0, \quad y(0) = 1, \quad y'(0) = 0. \tag{13}$$

Multiplying t and then taking the Laplace transform on both sides of Eq. (13) gives

$$-s^2\mathcal{L}'[y(t)] - y(0) + \mathcal{L}[ty^5(t)] = 0. \tag{14}$$

Operating with Laplace transform on both sides of Eq. (14) and applying by the same way proceeding as the Eqs. (3)-(9) we obtain the following recursive way

$$y_{n+1}(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_n(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_n(t)] - y_n(0) + \mathcal{L}[ty_n^5(t)]\}ds\right\}, \quad n \geq 0 \tag{15}$$

Let $y_0(t) = 1$, then, from Eq. (15), we have

$$y_1(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_0(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_0(t)] - y_0(0) + \mathcal{L}[ty_0^5(t)]\}\, ds\right\},$$
$$= 1 - \frac{1}{6}t^2,$$

In the same way the other iterations

$$y_2(t) = 1 - \frac{1}{6}t^2 + \frac{1}{24}t^4 - \frac{5}{756}t^6 + \frac{5}{7776}t^8 - \frac{1}{28512}t^{10} + \frac{1}{1213056}t^{12},$$
$$y_3(t) = 1 - \frac{1}{6}t^2 + \frac{1}{24}t^4 - \frac{5}{432}t^6 + \frac{55}{18144}t^8 - \frac{221}{299376}t^{10} + \frac{93437}{560431872}t^{12}$$
$$- \frac{638941}{18307441152}t^{14} + \frac{159505}{23538138624}t^{16} - O(t^{18}),$$
$$y_4(t) = 1 - \frac{1}{6}t^2 + \frac{1}{24}t^4 - \frac{5}{432}t^6 + \frac{35}{10368}t^8 - \frac{199}{199584}t^{10} + \frac{2609}{8895744}t^{12}$$
$$- \frac{501899}{5884534656}t^{14} + \frac{547728065}{22408307970048}t^{16} - \frac{3107584225}{450802430926848}t^{18}$$
$$+ \frac{325832483089}{170403318890348544}t^{20} - O(t^{22}).$$

This series has the closed form as $n \to \infty$ gives $\frac{1}{\sqrt{1+\frac{1}{3}t^2}}$, i.e.,

$$y_{Exact}(t) = \frac{1}{\sqrt{1 + \frac{1}{3}t^2}},$$

which is the exact solution of the problem 2.

In **Table 2** present the numerical results applying the LTVIM $(\boldsymbol{y_4(t)})$, the Padé approximation (PA) of order [4/4] with the exact solution $(y_{Exact}(t))$.

$$[4/4] = \frac{\sum_{i=0}^{4} a_i t^i}{1 + \sum_{i=1}^{4} b_i t^i} = \frac{\frac{1}{144}t^4 + \frac{1}{4}t^2 + 1}{\frac{5}{144}t^4 + \frac{5}{12}t^2 + 1}.$$

Table 2. Numerical results for problem 2

t	$y_{Exact}(t)$	$y_4(t)$	AE_1	PA [4/4]	AE_2
0.0	1.000000000	1.000000000	0.000E-00	1.000000000	0.000E-00
0.1	0.998337488	0.998337488	1.549E-15	0.998337488	7.956E-16
0.2	0.993399267	0.993399267	1.538E-12	0.993399267	7.909E-13
0.3	0.985329278	0.985329278	8.434E-11	0.985329278	4.342E-11
0.4	0.974354703	0.974354705	1.396E-09	0.974354704	7.205E-10
0.5	0.960768922	0.960768934	1.191E-08	0.960768928	6.161E-09
0.6	0.944911182	0.944911248	6.639E-08	0.944911216	3.445E-08
0.7	0.927145540	0.927145815	2.749E-07	0.927145684	1.431E-07
0.8	0.907841299	0.907842212	9.133E-07	0.907841776	4.775E-07
0.9	0.887356509	0.887359069	2.559E-06	0.887357853	1.343E-06
1.0	0.866025403	0.866031669	6.265E-06	0.866028708	3.304E-06

In Figure 2, a very good agreement is shown between the exact solution ($y_{Exact}(t)$) with a continuous line and the LTVIM $(\boldsymbol{y_4(t)})$ with the symbol o.

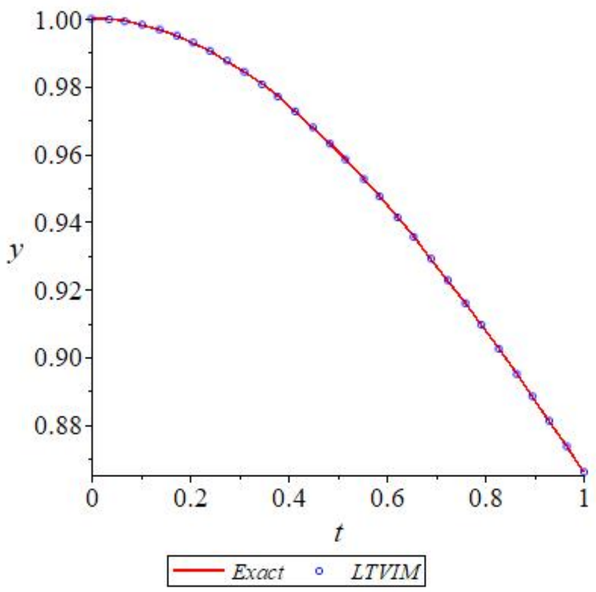

Fig. 2. Graph of $y_{Exact}(t)$ and LTVIM $(\boldsymbol{y_4(t)})$

Problem 3. Solve the following non-linear homogeneous Lane-Emden equation by using LTVIM

$$y''(t) + \frac{2}{t}y'(t) + 8e^{y(t)} + 4e^{y(t)/2} = 0, \quad y(0) = 0, \quad y'(0) = 0 \qquad (16)$$

Multiplying t and then taking the Laplace transform on both sides of Eq. (16) gives

$$-s^2\mathcal{L}'[y(t)] - y(0) + \mathcal{L}\left[8te^{y(t)} + 4te^{y(t)/2}\right] = 0. \qquad (17)$$

Operating with Laplace transform on both sides of Eq. (17) and applying by the same way proceeding as the Eqs. (3)-(9) we obtain the following recursive way

$$y_{n+1}(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_n(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_n(t)] - y_n(0) + \mathcal{L}\left[8te^{y_n(t)} + 4te^{y_n(t)/2}\right]\}ds\right\}, \qquad n \geq 0 \qquad (18)$$

Let $y_0(t) = 0$, then, from (18), we have

$y_1(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_0(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_0(t)] - y_0(0) + \mathcal{L}\left[8te^{y_0(t)} + 4te^{y_0(t)/2}\right]\}ds\right\},$
$\quad = -2t^2,$
$y_2(t) = -2t^2 + t^4 - \frac{3}{7}t^6 + \frac{17}{108}t^8 - \frac{1}{20}t^{10},$
$y_3(t) = -2t^2 + t^4 - \frac{2}{3}t^6 + \frac{353}{756}t^8 - \frac{1247}{3780}t^{10} + \frac{1427}{6552}t^{12} - O(t^{14}).$
$y_4(t) = -2t^2 + t^4 - \frac{2}{3}t^6 + \frac{1}{8}t^8 - \frac{16507}{41580}t^{10} + \frac{2621}{8424}t^{12} - \frac{355139}{1474200}t^{14} + O(t^{16}).$
This series has the closed form as $n \to \infty$ gives $-2ln(t^2+1)$, i.e.,
$y_{Exact}(t) = -2ln(t^2+1),$
which is the exact solution of the problem 3.
In **Table 3** present the numerical results applying the LTVIM $(\boldsymbol{y_4(t)})$, the Padé approximation (PA) of order [4/4] with the exact solution $(y_{Exact}(t))$.
$[4/4] = \frac{\sum_{i=0}^{4} a_i t^i}{1 + \sum_{i=1}^{4} b_i t^i} = \frac{-t^4 - 2t^2}{\frac{1}{6}t^4 + t^2 + 1}.$

Table 3. Numerical results for problem 3

t	$y_{Exact}(t)$	$y_4(t)$	AE_1	PA [4/4]	AE_2
0.0	0.0	0.0	0.0	0.0	0.0
0.1	-0.019900661	-0.019900661	2.788E-13	-0.019900661	1.083E-12
0.2	-0.078441426	-0.078441426	2.238E-10	-0.078441425	1.030E-09
0.3	-0.172355392	-0.172355384	7.842E-09	-0.172355339	5.276E-08
0.4	-0.296840010	-0.296839970	3.958E-08	-0.296839212	7.981E-07
0.5	-0.446287102	-0.446287611	5.090E-07	-0.446280991	6.110E-06
0.6	-0.614969399	-0.614977084	7.684E-06	-0.614939200	3.019E-05
0.7	-0.797552239	-0.797603866	5.162E-05	-0.797442293	1.099E-04
0.8	-0.989392483	-0.989627708	2.352E-04	-0.989072744	3.197E-04
0.9	-1.186653690	-1.187485463	8.317E-04	-1.185870216	7.834E-04
1	-1.386294361	-1.388743062	2.448E-03	-1.384615384	1.678E-03

In Figure 3, a very good agreement is shown between the exact solution $(y_{Exact}(t))$ with a continuous line and the LTVIM $(\boldsymbol{y_4(t)})$ with the symbol o.

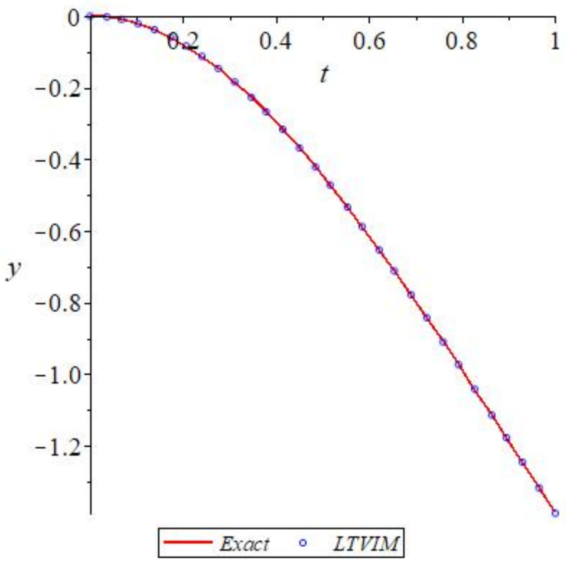

Fig. 3. Graph of $y_{Exact}(t)$ and LTVIM $(\boldsymbol{y_4(t)})$

3.2 LANE–EMDEN_LINEAR BVPS

Problem 4. Solve the following linear homogeneous Lane-Emden equation by using LTVIM

$$y'' + \frac{2}{t}y' - (4t^2 + 6)y = 0, \quad y(0) = 1, \quad y(1) = e \tag{19}$$

Multiplying t and then taking the Laplace transform on both sides of Eq. (19) gives

$$-s^2\mathcal{L}'[y(t)] - y(0) - \mathcal{L}[(4t^3 + 6t)y(t)] = 0. \tag{20}$$

Operating with Laplace transform on both sides of Eq. (20) and applying by the same way proceeding as the Eqs. (3)-(9) we obtain the following recursive way

$$y_{n+1}(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_n(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_n(t)] - y_n(0) - \mathcal{L}[(4t^3 + 6t)y_n(t)]\}ds\right\}, \qquad n \geq 0 \tag{21}$$

Let $y_0(t) = 1 + tC$, then, from (21), we have

$$y_1(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_0(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_0(t)] - y_0(0) - \mathcal{L}[(4t^3 + 6t)y_0(t)]\}ds\right\},$$

$$= 1 + t^2 + \frac{1}{2}Ct^3 + \frac{1}{5}t^4 + \frac{2}{15}Ct^5,$$

matching $y_1(t)$ with the boundary condition $y(1) = e$, we obtain $C = 0.818339729$ and then

$$y_1(t) = 1 + t^2 + 0.409169864t^3 + 0.2000000000t^4 + 0.109111963t^5$$

$$y_2(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_1(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_1(t)] - y_1(0) - \mathcal{L}[(4t^3 + 6t)y_1(t)]\}ds\right\},$$

$$= 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{10}Ct^5 + \frac{13}{105}t^6 + \frac{1}{20}Ct^7 + \frac{1}{90}t^8 + \frac{4}{675}Ct^9,$$

matching $y_2(t)$ with the boundary condition $y(1) = e$, we get $C = 0.534620481$ and hence

$$y_2(t) = 1 + t^2 + \frac{1}{2}t^4 + 0.053462048t^5 + 0.123809523t^6 + 0.0267310240t^7$$
$$+0.011111111t^8 + 0.003168121t^9.$$

In the same way the other iterations

$$y_3(t) = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + 0.004266850t^7 + 0.038095238t^8 + 0.003097417t^9$$
$$+0.005108225t^{10} + O(0.000710662t^{11}),$$

$$y_4(t) = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{1}{24}t^8 + 0.000227207t^9 + \frac{47}{5775}t^{10} + 0.000215732t^{11}$$
$$+O\left(\frac{151}{128700}t^{12}\right),$$

$$y_5(t) = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{1}{24}t^8 + \frac{1}{120}t^{10} + 0.000008611t^{11} + \frac{2489}{1801800}t^{12}$$
$$+O(t^{13}).$$

This series has the closed form as $n \to \infty$ gives e^{t^2}, i.e.,

$$y_{Exact}(t) = e^{t^2},$$

which is the exact solution of the problem 4.

In **Table 4** present the numerical results applying the LTVIM $(y_5(t))$, the Padé approximation (PA) of order [8/8] with the exact solution $(y_{Exact}(t))$.

$$[8/8] = \frac{\sum_{i=0}^{8} a_i x^i}{1 + \sum_{i=1}^{8} b_i x^i} = \frac{p_8(t)}{q_8(t)}$$

where

$$p_8(t) = 1 - 0.116078680t + 0.613834217t^2 - 0.0604386944t^3$$
$$+0.166525406t^4 - 0.0127042126t^5 + 0.0244189992t^6$$
$$-0.00111560230t^7 + 0.00169835758t^8,$$

$$q_8(t) = 1 - 0.116078680t - 0.386165782t^2 + 0.0556399863t^3$$
$$+0.0526911894t^4 - 0.0103048586t^5 - 0.00185596540t^6$$
$$+0.000715709972t^7 - 0.0000969745862t^8.$$

Table 4. Numerical results for problem 4

t	$y_{Exact}(t)$	$y_5(t)$	AE_1	PA [8/8]	AE_2
0.0	1.000000000	1.000000000	0.000E-00	1.000000000	0.000E-00
0.1	1.010050167	1.010050167	7.96E-17	1.010050167	7.960E-17
0.2	1.040810774	1.040810774	1.524E-13	1.040810774	1.524E-13

0.3	1.094174283	1.094174283	1.245E-11	1.094174283	1.245E-11
0.4	1.173510870	1.173510871	2.793E-10	1.173510871	2.794E-10
0.5	1.284025416	1.284025419	3.065E-09	1.284025419	3.066E-09
0.6	1.433329414	1.433329435	2.104E-08	1.433329435	2.106E-08
0.7	1.632316219	1.632316321	1.013E-07	1.632316321	1.015E-07
0.8	1.896480879	1.896481231	3.519E-07	1.896481232	3.530E-07
0.9	2.247907986	2.247908770	7.834E-07	2.247908767	7.812E-07
1.0	2.718281828	2.718281828	1.000E-19	2.718281734	9.390E-08

In Figure 4, a very good agreement is shown between the exact solution $(y_{Exact}(t))$ with a continuous line and the LTVIM $(\boldsymbol{y_5(t)})$ with the symbol o.

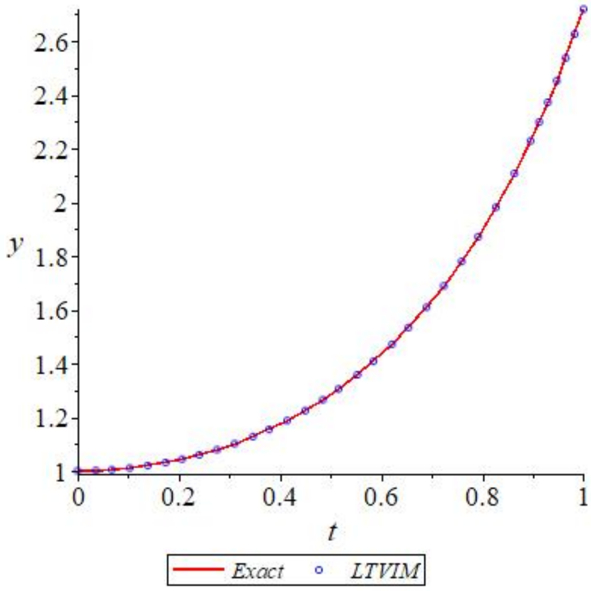

Fig. 4. Graph of $y_{Exact}(t)$ and LTVIM $(\boldsymbol{y_5(t)})$

Problem 5. Solve the following linear nonhomogeneous Lane-Emden equation by using LTVIM

$$y'' + \frac{1}{t}y' - y - (4 + 9t - t^2 - t^3) = 0, \quad y(0) = 0, \qquad y(1) = 2 \tag{22}$$

Multiplying t and then taking the Laplace transform on both sides of Eq. (22) gives

$$(-s^2)\mathcal{L}'[y(t)] - sY(s) - \mathcal{L}[ty(t)] - \left(\frac{4}{s^2} + \frac{18}{s^3} - \frac{6}{s^4} - \frac{24}{s^5}\right) = 0. \tag{23}$$

Operating with Laplace transform on both sides of (23) and applying by the same way proceeding as the Eqs. (3)-(9) we obtain the following recursive way

$$y_{n+1}(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_n(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_n(t)] - s\mathcal{L}[y_n(t)] - \mathcal{L}[ty_n(t)] - \left(\frac{4}{s^2} + \frac{18}{s^3} - \frac{6}{s^4} - \frac{24}{s^5}\right)\right\} ds\Bigg\}, \quad n \geq 0 \tag{24}$$

Let $y_0(t) = tC$, then, from (24), we have

$$y_1(t) = \mathcal{L}^{-1}\left\{\mathcal{L}[y_0(t)] + \int \frac{1}{s^2}\{-s^2\mathcal{L}'[y_0(t)] - s\mathcal{L}[y_0(t)] - \mathcal{L}[ty_0(t)] - \left(\frac{4}{s^2} + \frac{18}{s^3} - \frac{6}{s^4} - \frac{24}{s^5}\right)\right\} ds\Bigg\},$$

$$= \frac{1}{2}Ct + \frac{2}{3}t^2 + \frac{1}{12}Ct^3 + \frac{3}{4}t^3 - \frac{1}{20}t^4 - \frac{1}{30}t^5,$$

matching $y_1(t)$ with the boundary condition $y(1) = 2$, getting $C = 1.142857142$ and hence

$$y_1(t) = \frac{4}{7}t + \frac{2}{3}t^2 + \frac{71}{84}t^3 - \frac{1}{20}t^4 - \frac{1}{30}t^5,$$

$$y_2(t)=\mathcal{L}^{-1}\Big\{\mathcal{L}[y_1(t)]+\int\frac{1}{s^2}\{-s^2\mathcal{L}'[y_1(t)]-s\mathcal{L}[y_1(t)]$$
$$-\mathcal{L}[ty_1(t)]-\left(\frac{4}{s^2}+\frac{18}{s^3}-\frac{6}{s^4}-\frac{24}{s^5}\right)\Big\}ds\Big\},$$
$$=\frac{1}{4}Ct+\frac{8}{9}t^2+\frac{1}{16}Ct^3+\frac{15}{16}t^3-\frac{2}{75}t^4-\frac{1}{72}t^5+O\left(\frac{1}{360}Ct^5\right),$$

matching $y_2(t)$ with the boundary condition $y(1)=2$, obtaining $C=0.684959093$ and hence

$$y_2(t)=\frac{2721}{15890}t+\frac{8}{9}t^2+\frac{124617}{127120}t^3-\frac{2}{75}t^4-O\left(\frac{34283}{2860200}t^5\right).$$

In the same way the other iterations

$$y_3(t)=\mathcal{L}^{-1}\Big\{\mathcal{L}[y_2(t)]+\int\frac{1}{s^2}\{-s^2\mathcal{L}'[y_2(t)]-s\mathcal{L}[y_2(t)]$$
$$-\mathcal{L}[ty_2(t)]-\left(\frac{4}{s^2}+\frac{18}{s^3}-\frac{6}{s^4}-\frac{24}{s^5}\right)\Big\}ds\Big\},$$
$$=\frac{1462701}{27781600}t+\frac{26}{27}t^2+\frac{31741867}{31750400}t^3-\frac{49}{4500}t^4-\frac{4535599}{1363824000}t^5-O\left(\frac{71}{88200}t^6\right),$$
$$y_4(t)=\frac{12936071389}{786028874400}t+\frac{80}{81}t^2+\frac{2518403108539}{2515292398080}t^3-\frac{68}{16875}t^4-\frac{418735246837}{509346710611200}t^5$$
$$-\frac{1733}{4630500}t^6-O\left(\frac{50027789653}{475390263237120}t^7\right),$$
$$y_5(t)=0.00520766151t+0.995884774t^2+1.00070508t^3-0.00142320988t^4$$
$$-0.185151473\times10^{-3}t^5-0.149408940\times10^{-3}t^6-0.312898079\times10^{-4}t^7$$
$$-O(0.664441752\times10^{-5}t^8),$$
$$y_6(t)=0.00166421435t+0.998628258t^2+1.00030193t^3-0.490403292\times10^{-3}t^4$$
$$-0.368478762\times10^{-4}t^5-0.552300837\times10^{-4}t^6-0.847872222\times10^{-5}t^7$$
$$-0.281339278\times10^{-5}t^8-O(0.539325814\times10^{-8}t^9).$$

This series has the closed form as $n\to\infty$ gives t^2+t^3, i.e.,

$y_{Exact}(t)=t^2+t^3,$

which is the exact solution of the problem 5.

In **Table 5** present the numerical results applying the LTVIM $(\boldsymbol{y_6(t)})$, the Padé approximation (PA) of order [4/4] with the exact solution $(y_{Exact}(t))$.

$$[4/4]=\frac{\sum_{i=0}^{4}a_it^i}{1+\sum_{i=1}^{4}b_it^i}$$
$$=\frac{0.0016642143t+0.998078524t^2+0.670428755t^3-0.331093495t^4}{1-0.330326000t-0.000177687t^2+0.000052684t^3-0.000009742t^4}.$$

Table 5. Numerical results for problem 5

$\boldsymbol{t}$	$\boldsymbol{y_{Exact}(t)}$	$\boldsymbol{y_6(t)}$	$\boldsymbol{AE_1}$	**PA [4/4]**	$\boldsymbol{AE_2}$
0.0	0.000000000	0.000000000	0.000E-00	0.000000000	0.000E-00
0.1	0.011	0.011152956	1.529 E-04	0.011152956	1.529E-04
0.2	0.048	0.048279588	2.795E-04	0.048279588	2.795E-04
0.3	0.117	0.117379855	3.798E-04	0.117379855	3.798E-04
0.4	0.224	0.224452356	4.523E-04	0.224452356	4.523E-04
0.5	0.375	0.375494169	4.941E-04	0.375494168	4.941E-04
0.6	0.576	0.576500629	5.006E-04	0.576500623	5.006E-04
0.7	0.833	0.833465037	4.650E-04	0.833465012	4.650E-04
0.8	1.152	1.152378291	3.782E-04	1.152378203	3.782E-04
0.9	1.539	1.539228419	2.284E-04	1.539228146	2.281E-04
1.0	2.000	2.000000000	2.370E-23	1.999999241	7.587E-07

In Figure 5, a very good agreement is shown between the exact solution $(y_{Exact}(t))$ with a continuous line and the LTVIM $(\boldsymbol{y_6(t)})$ with the symbol o.

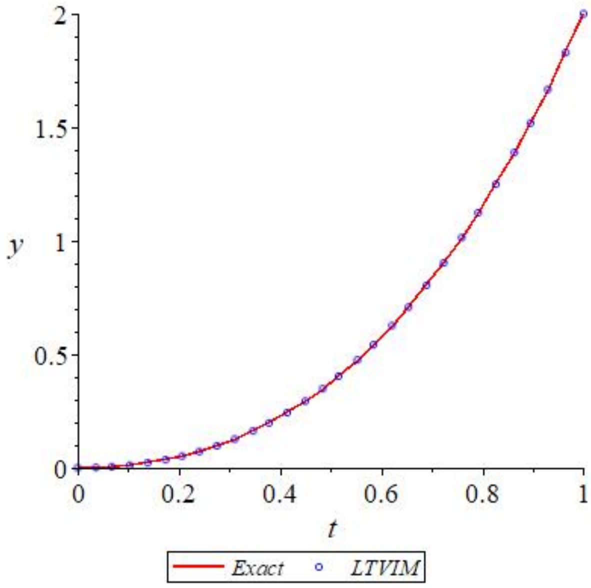

Fig. 5. Graph of $y_{Exact}(t)$ and LTVIM $\left(\boldsymbol{y_6(t)}\right)$

4. CONCLUSIONS

Five problems IVPs and BVPs from second order of Lane-Emden equation were successfully solved by applying the strategy LTVIM. The proposed method showed fast converge towards exact solution of the solved problems in this paper and thus can be applied on other equations with differential order and obtain a solution that converges to the exact solution, in addition the method succeeded in dealing with the equation despite the presence of the singular behaviour point, The LTVIM wider variety of applications stems from its adept handling of various types , The LTVIM idea has been used directly without the requirement for restricting assumptions or transformation formulas , also the approximate solution we obtained is more accurate when the value of n increase.

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CONFLICTS OF INTEREST

The author declares no conflict of interest.

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