

Non-Compatible Action Graph and Its Adjacency Matrix for The Non-abelian Tensor Product for Groups of Prime Power Order

Mohammed Khalid Shahoodh¹ 

¹Ministry of Education, General Directorate of Education in Ramadi, Al-idrisi Secondary School for Girls, Anbar, Iraq

*Corresponding Author: Mohammed Khalid Shahoodh

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ABSTRACT: This article focused on the notion of the non-abelian tensor product of groups of prime power order. Particularly, it presented new graph named as Non-compatible action graph and discussed some of its properties. Moreover, this graph concentrated on the case of non-compatible actions of the tensor product of two finite \mathfrak{p} -groups. Furthermore, its adjacency matrix has been determined and discussed in detail. Moreover, the adjacency matrix has been denoted by $\hat{A}(\Gamma_{\mathfrak{U} \otimes \mathfrak{U}'}^{non})$ and its inputs are 1 whenever there is adjacency and 0 otherwise.

Keywords: Non-abelian tensor product, Graph theory, \mathfrak{p} -groups, Action, Compatible actions.



1. INTRODUCTION

To construct new graph from a specific algebraic structure, we need to consider its structure with its properties. For instance, groups and rings are different algebraic structures with different conditions both having the commutativity property. Based on this property, there are some graphs have been introduced and named by commuting and non-commuting graphs. More specifically, in [1] the author studied a graph named by commuting graph for a finite group and in such graph any two different vertices are connect whenever the product of them is abelian. Moreover, the other side of the product has been provided by [2] as a non-commuting graph of a finite groups and any two distinct vertices are adjacent whenever their product is non-abelian. By following this approach, many authors followed this direction by introducing new graphs each one is distinct from the other according to the property that has been considered by the given graph. Some of these works was the conjugacy class graph of a given group which been provided by Bertram et.al [3]. This graph was defined when the vertices set of it which is the set of non-central conjugacy classes of a group and two different vertices are adjacent whenever their cardinalities are not co-prime. Furthermore, a graph named by neutrosophic graph for some finite groups has been presented by Chalapath and Kumar [4] with some of its basic properties have been investigated. In [5] the author introduced the compatible action graph for groups of prime power order which focused on the case of compatibility of actions when \mathfrak{p} is an odd prime. Then, in [6] he completed the sub-graph of the mentioned graph with some of its properties. In this paper, we introduced the non-compatible action graph of two finite groups of prime power order by considering the case of non-compatible actions of their non-abelian tensor product and studied some properties of it. Then, we computed its adjacency matrix with some of its properties.

2. BASIC CONCEPTS

This section included some definitions and some of the past results which are used in this study which have been stated as follows.

Definition 2.1 [7] Let \mathfrak{U} and \mathfrak{U}' are \mathfrak{p} -groups. Then, an action of \mathfrak{U} on \mathfrak{U}' is the mapping $\varphi: \mathfrak{U} \rightarrow \text{Aut}(\mathfrak{U}')$ in which $\varphi(xy)(z) = \varphi(x)(\varphi(y)(z))$ where $x, y \in \mathfrak{U}$ and $z \in \mathfrak{U}'$.

Definition 2.2 [7] Let \bar{U} and \bar{U}' are \mathfrak{p} -groups in which each one act on the other. The actions of these groups are said to be compatible on each other with the actions of \bar{U} and \bar{U}' on themselves by conjugation if ${}^x(\mathfrak{y})x' = {}^x(\mathfrak{y}(x^{-1}x'))$ and ${}^{\mathfrak{y}}x = \mathfrak{y}(x(\mathfrak{y}^{-1}x'))$ for all $x, x' \in \bar{U}$ and $\mathfrak{y}, \mathfrak{y}' \in \bar{U}'$.

Proposition 2.1 [7] Let \bar{U} and \bar{U}' be two groups such that the action of \bar{U} on \bar{U}' is trivial. If \bar{U} is abelian, then the action of \bar{U} is compatible with any other action of \bar{U}' .

Theorem 2.1 [8] Let $\bar{U} = \langle x \rangle \cong \check{C}_{\mathfrak{p}^m}$ and $\bar{U}' = \langle \mathfrak{y} \rangle \cong \check{C}_{\mathfrak{p}^n}$ are groups of prime power order where \mathfrak{p} is an odd prime and $m, n \geq 2$. Moreover, let $\eta \in \text{Aut}(\bar{U})$ for which $|\eta| = \mathfrak{p}^\omega, \omega = 1, \dots, m - 1$ and $\eta' \in \text{Aut}(\bar{U}')$ in which $|\eta'| = \mathfrak{p}^\varepsilon, \varepsilon = 1, \dots, n - 1$. Then, (η, η') is compatible pair iff $\omega + \varepsilon \leq \min \{m, n\}$.

Theorem 2.2 [8] Let $\bar{U} = \langle x \rangle \cong \check{C}_{2^m}$ and $\bar{U}' = \langle \mathfrak{y} \rangle \cong \check{C}_{2^n}$ are groups of 2-power order in which the actions upon each other given by ${}^x\mathfrak{y} = \mathfrak{y}^u$ and ${}^{\mathfrak{y}}x = x^v$ with $u, v \in \mathbb{N}$. Moreover, let the action of \bar{U}' on \bar{U} has an order 2^s and the action of \bar{U} on \bar{U}' has an order 2^t with $s, t \in \mathbb{N}$. Then, the action are compatible iff $v \equiv 1 \pmod{2^s}$ and $u \equiv 1 \pmod{2^t}$.

Theorem 2.3 [9] Let \bar{U} be a cyclic group of order $2^m, m \geq 3$. Then, $\text{Aut}(\bar{U}) \cong \check{C}_2 \times \check{C}_{2^{m-2}}$ and $|\text{Aut}(\bar{U})| = 2^{m-1}$.

Theorem 2.4 [9] Let \bar{U} be a cyclic group of prime power order with \mathfrak{p} is an odd prime and $m \in \mathbb{Z}^+$. Then, $\text{Aut}(\bar{U}) \cong \check{C}_{\mathfrak{p}-1} \times \check{C}_{\mathfrak{p}^{m-1}}$ and $|\text{Aut}(\bar{U})| = (\mathfrak{p} - 1)\mathfrak{p}^{m-1}$.

Definition 2.3 [10] Let $\hat{A} = [a_{ij}]$ be a square matrix. The trace of \hat{A} is the sum of its elements which are on the main diagonal of \hat{A} .

The paragraph bellow, contains some basic definitions of graph theory which are important in completing the results of the present study. These basic definitions have been cited from [11-12].

Suppose \bar{U} is a graph, then \bar{U} is connected if any two different vertices of \bar{U} having a path between them. Let \bar{U} be a connected graph, if any vertex of \bar{U} has an even degree, then it's called an Eulerian graph. Furthermore, if \bar{U} has solely two vertices of odd degree, then it's called semi-Eulerian. Moreover, by regular graph we mean a graph all of its vertices having the same degree. A graph \bar{U} is said to be complete graph if there is only one edge between each pair of its vertices. Finally, a graph \bar{U} is said to be bipartite graph if its vertices set has the property that can be partitioned into two sets V_1 and V_2 such that they are disjoint and any edge in \bar{U} links a vertex from V_1 with the other from V_2 .

3. MAIN RESULTS

This section contains the main results of this paper which started with the following definition.

Definition 3.1 Let \bar{U} and \bar{U}' be two \mathfrak{p} -groups and let $\bar{U} \otimes \bar{U}'$ be a non-abelian tensor product of \bar{U} and \bar{U}' with (η, η') be a pair of non-compatible actions of $\bar{U} \otimes \bar{U}'$. Then, $\Gamma_{\bar{U} \otimes \bar{U}'}^{non}$ is a non-compatible action graph consisting of $V(\Gamma_{\bar{U} \otimes \bar{U}'}^{non})$ which is the set of $\text{Aut}(\bar{U})$ and $\text{Aut}(\bar{U}')$ and $\hat{E}(\Gamma_{\bar{U} \otimes \bar{U}'}^{non})$ which is the set contains of all non-compatible pair of actions. Furthermore, η and η' are connect whenever they are not compatible.

Theorem 3.1 Let $\bar{U} = \langle x \rangle \cong \check{C}_{\mathfrak{p}^m}$ and $\bar{U}' = \langle \mathfrak{y} \rangle \cong \check{C}_{\mathfrak{p}^n}$ are \mathfrak{p} -groups. Then, the order of $\Gamma_{\bar{U} \otimes \bar{U}'}^{non}$ is given as follows.

- 1) If $\mathfrak{p} = 2, m, n \geq 4$ with $\bar{U} = \bar{U}'$, then the order is 2^{m-1} .
- 2) If $\mathfrak{p} = 2, m, n \geq 4$ with $\bar{U} \neq \bar{U}'$, then the order is $2^{m-1} + 2^{n-1}$.
- 3) If \mathfrak{p} is an odd prime with $m = n \geq 3$, then the order is $(\mathfrak{p} - 1)\mathfrak{p}^{m-1}$.
- 4) If \mathfrak{p} is an odd prime with $m \neq n \geq 3$, then the order is $(\mathfrak{p} - 1)(\mathfrak{p}^{m-1} + \mathfrak{p}^{n-1})$.

Proof: It follows by Definition 3.1 and Theorems 2.1, 2.2. ■

Theorem 3.2 Let $\bar{U} = \langle x \rangle \cong \check{C}_{\mathfrak{p}^m}$ and $\bar{U}' = \langle \mathfrak{y} \rangle \cong \check{C}_{\mathfrak{p}^n}$ are \mathfrak{p} -groups. Then, $\Gamma_{\bar{U} \otimes \bar{U}'}^{non}$ is not connected graph.

Proof: Since our dealing with \mathfrak{p} -groups, then we have to consider two cases. The first one is when $\mathfrak{p} = 2$ and the second is when \mathfrak{p} is an odd prime. Now, let $\mathfrak{p} = 2$ with $m, n \geq 4$. Furthermore, let $\eta \in \text{Aut}(\bar{U})$ and $\eta' \in \text{Aut}(\bar{U}')$ be two vertices of $V(\Gamma_{\bar{U} \otimes \bar{U}'}^{non})$, then if $|\eta| = 1$, then by Proposition 2.1, η is compatible with any other vertex of $\text{Aut}(\bar{U}')$. But

Definition 3.1 gives us that η is not connect with any $\eta' \in \text{Aut}(U')$. This implies that $\Gamma_{U \otimes U'}^{non}$ is not connected. Finally, let $|\eta| = 2^m$ with m is positive integer. Then, by Theorem 2.2, η is compatible with any η' that have 2-power order. Thereby, Definition 3.1, forces that η and η' are not adjacent. Therefore, $\Gamma_{U \otimes U'}^{non}$ is not connected graph. For the second case, we can use the same way to prove the result by using Definition 3.1, Proposition 2.1 and Theorem 2.1. ■

Theorem 3.3 Let $U = \langle x \rangle \cong \check{C}_{p^m}$ and $U' = \langle y \rangle \cong \check{C}_{p^n}$ are p -groups. Then, $\Gamma_{U \otimes U'}^{non}$ is a bipartite graph iff $U \neq U'$.

Proof: As we considered p -groups, then the proof will divided into two parts. The first part is when p is an even prime and the second part is when p is an odd prime. Regarding the first part, let $p = 2$ with $m, n \geq 4$ and let $\Gamma_{U \otimes U'}^{non}$ be a bipartite graph. That is mean $V(\Gamma_{U \otimes U'}^{non})$ can be partitioned into two sets, say V_1 and V_2 for which $V_1 \cap V_2 = \Phi$. Now, let $U = U'$, then $\text{Aut}(U) = \text{Aut}(U')$. This gives us that, there are some vertices makes a loops with themselves and this forces that $V(\Gamma_{U \otimes U'}^{non})$ cannot be partitioned. Thereby, we got a contradiction. Hence, $U \neq U'$. Conversely, let $U \neq U'$, then $\text{Aut}(U) \neq \text{Aut}(U')$. This implies that $V(\Gamma_{U \otimes U'}^{non})$ can be partitioned into two disjoint sets $\text{Aut}(U)$ and $\text{Aut}(U')$ in which a vertex from $\text{Aut}(U)$ connect a vertex from $\text{Aut}(U')$. Therefore, $\Gamma_{U \otimes U'}^{non}$ is a bipartite graph. By similar manner, we can prove the second part. ■

Corollary 3.1 Let $U = \langle x \rangle \cong \check{C}_{2^m}$ and $U' = \langle y \rangle \cong \check{C}_{2^n}$ are groups of 2-power order with $m, n \geq 4$. Then,

1. $\Gamma_{U \otimes U'}^{non}$ is not Eulerian (resp. semi-Eulerian) graph.
2. $\Gamma_{U \otimes U'}^{non}$ is not regular graph.
3. $\Gamma_{U \otimes U'}^{non}$ is not complete graph.

Proof: Clear. ■

Corollary 3.2 Let $U = \langle x \rangle \cong \check{C}_{p^m}$ and $U' = \langle y \rangle \cong \check{C}_{p^n}$ are groups of prime power order with p is an odd prime and $m, n \geq 3$. Then,

1. $\Gamma_{U \otimes U'}^{non}$ is not Eulerian (resp. semi-Eulerian) graph.
2. $\Gamma_{U \otimes U'}^{non}$ is not regular graph.
3. $\Gamma_{U \otimes U'}^{non}$ is not complete graph.

Proof: Clear. ■

In the next results, we discussed the adjacency matrix of this graph with some properties of it. We started with the following proposition.

Proposition 3.1 Let $U = \langle x \rangle \cong \check{C}_{2^4}$ be a p -group with $p = 2$. Then, $\hat{A}(\Gamma_{U \otimes U}^{non})$ is given bellow.

$$\hat{A}(\Gamma_{U \otimes U}^{non}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Proof: Clear by Definition 3.1 and Theorem 3.1 point one.

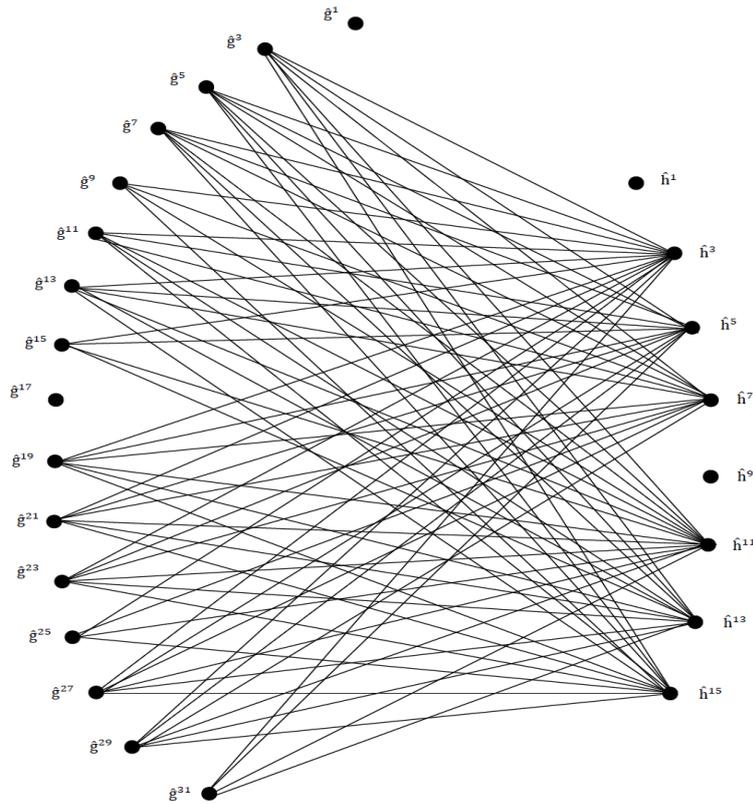


FIGURE 2. Non-Compatible Action Graph for Proposition 3.2.

Theorem 3.4 Let $\check{U} = \langle x \rangle \cong \check{C}_{2^m}$ be a \mathfrak{p} -group with $\mathfrak{p} = 2$ and $m \geq 4$. Then, $\hat{A}(\Gamma_{\check{U} \otimes \check{U}}^{non})$ is a square matrix of size $2^{m-1} \times 2^{m-1}$.

Proof: Let $\eta, \eta' \in V(\Gamma_{\check{U} \otimes \check{U}}^{non})$, then by Definition 3.1, they are connect whenever they are not compatible. By Theorem 2.3, there are 2^{m-1} vertices in $V(\Gamma_{\check{U} \otimes \check{U}}^{non})$ which can be represented as rows and columns in $\hat{A}(\Gamma_{\check{U} \otimes \check{U}}^{non})$. Therefore, we achieved the result. ■

Theorem 3.5 Let $\check{U} = \langle x \rangle \cong \check{C}_{2^m}$ and $\check{U}' = \langle y \rangle \cong \check{C}_{2^n}$ are groups of 2-power order with $m, n \geq 4$. Then, $\hat{A}(\Gamma_{\check{U} \otimes \check{U}'}^{non})$ is a square matrix of size $(2^{m-1} + 2^{n-1}) \times (2^{m-1} + 2^{n-1})$.

Proof: It follows by Theorem 3.4. ■

Proposition 3.3 Let $\check{U} = \langle x \rangle \cong \check{C}_{3^3}$ be a \mathfrak{p} -group with $\mathfrak{p} = 3$. Then, $\hat{A}(\Gamma_{\check{U} \otimes \check{U}}^{non})$ is given bellow.

$$\hat{A}(\Gamma_{\mathfrak{U} \otimes \mathfrak{U}}^{non}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Proof: Clear by Definition 3.1 and Theorem 3.1 point three. ■

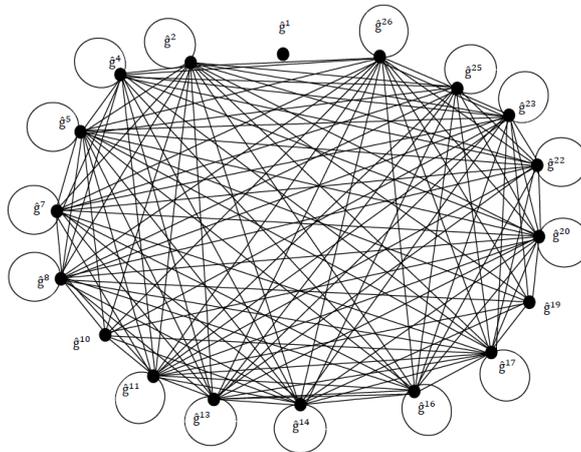


FIGURE 3. Non-Compatible Action Graph for Proposition 3.3.

Proposition 3.4 Let $\mathfrak{U} = \langle x \rangle \cong \check{C}_{3^3}$ and $\mathfrak{U}' = \langle y \rangle \cong \check{C}_{3^4}$ are \mathfrak{p} -groups with $\mathfrak{p} = 3$. Then, $\hat{A}(\Gamma_{\mathfrak{U} \otimes \mathfrak{U}'}^{non})$ is given below.

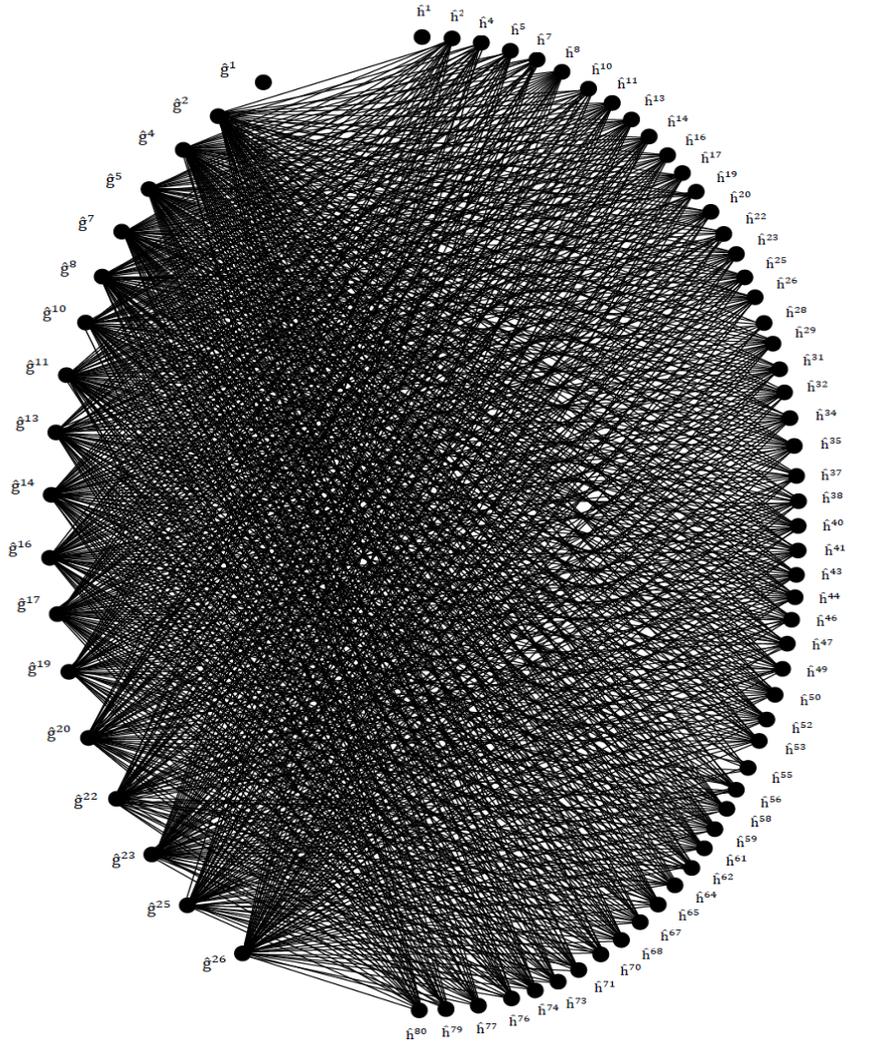


FIGURE 4. Non-Compatible Action Graph for Proposition 3.4.

Theorem 3.6 Let $\mathcal{U} = \langle x \rangle \cong \check{C}_{p^m}$ be a p -group with p is an odd prime and $m \geq 3$. Then, $\hat{A}(\Gamma_{\mathcal{U} \otimes \mathcal{U}}^{non})$ is a square matrix of size $((p - 1)p^{m-1}) \times ((p - 1)p^{m-1})$.

Proof: Let $\eta, \eta' \in V(\Gamma_{\mathcal{U} \otimes \mathcal{U}}^{non})$, then by Definition 3.1, they are adjacent if they are not compatible. By Theorem 2.4, there are $(p - 1)p^{m-1}$ vertices in $V(\Gamma_{\mathcal{U} \otimes \mathcal{U}}^{non})$ which can be represented as entries in $\hat{A}(\Gamma_{\mathcal{U} \otimes \mathcal{U}}^{non})$. Therefore, we obtained the result. ■

Theorem 3.7 Let $\mathcal{U} = \langle x \rangle \cong \check{C}_{p^m}$ and $\mathcal{U}' = \langle y \rangle \cong \check{C}_{p^n}$ are p -groups with p is an odd prime and $m, n \geq 3$. Then, $\hat{A}(\Gamma_{\mathcal{U} \otimes \mathcal{U}'}^{non})$ is a square matrix of size $((p - 1)(p^{m-1} + p^{n-1})) \times ((p - 1)(p^{m-1} + p^{n-1}))$.

Proof: It follows by Theorem 3.6. ■

Corollary 3.3 The trace of $\hat{A}(\Gamma_{\mathcal{U} \otimes \mathcal{U}'}^{non})$ is greater than or equal to zero in both cases when $\mathcal{U} = \mathcal{U}'$ and $\mathcal{U} \neq \mathcal{U}'$.

4. CONCLUSIONS

As a result, new graph has been introduced named as non-compatible action graph and some of its properties have been discussed. The results showed that this graph is not connected graph, not regular graph and its bipartite graph whenever the groups are not the same. Based on this study, many results can be extended to provide new properties of this graph. Furthermore, the adjacency matrix of this graph has been determined. The obtained results shown that $\hat{A}(\Gamma_{\bar{U} \otimes \bar{U}'}^{non})$ is a square matrix in the case of $\bar{U} = \bar{U}'$ and $\bar{U} \neq \bar{U}'$.

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CONFLICTS OF INTEREST

The author declares no conflict of interest.

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