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Generalized Projective Product of Semi-rings

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ABSTRACT: The concept of Differential algebra has been played an influential role in various directions of abstract algebra. This notion has been considered before fifty years ago with semi-ring and several types of rings. Furthermore, the commutatively of a given ring or semi-ring has been also investigated under the notion of derivation and many studies have been published in this way. Based on the concept of derivation, its generalization has been introduced by Bresar [9] and a lot of works are presented about this generalization in various types of rings and semi-rings. Motivated by these works in the literature, we introduced the concept of generalized projective product of n-number of semi-rings. In addition, we studied the convers of this case which is saying that if the generalized projective product of n-number of semi-rings existed, then there exist a corresponding derivations, generalized derivations, semi-derivations, generalized semi-derivations, Jordan derivations, generalized Jordan derivations, Jordan semi-derivations and Generalized Jordan semi-derivations on the semi-rings $R_i$respectively for any i=1,⋯,τ..

Keywords: Projective product, Semi-ring, Derivation, Generalized Derivation, Semi-derivation.

1. INTRODUCTION

A semi-ring is a mathematical system $(\mathcal{R},+,\cdot)$ with the binary operations $+$ and $\cdot$ such that $\cdot$ is distributive on $+$ for any element of $\mathcal{R}$. Many studies have been considered the concept of differential algebra with various algebraic structures in order to study some of its basic properties. For instance, Haetinger and Mamouni [1] introduced the concept of generalized left semi-derivations in the non-commutative ring and presented some results about it. A paper by Lee and Zhou [2] had studied the notion of Jordan $*$-derivations of prime rings. The authors showed that any Jordan $*$-derivation of non-commutative prime ring with involution $*$ is X-inner. In [3] Jing and Lu studied the generalized Jordan derivations and generalized Jordan triple derivations of prime rings. Particularly, they showed that any generalized Jordan derivations (resp. generalized Jordan triple derivations) of 2-torsion free prime ring is a generalized derivation. The idea of projective product of two semi-rings has been introduced by Sindhu et. al [4]. The concept of left semi-derivations has been discussed on prime near-rings by Bharathi and Ganesh [5]. They showed that the near-rings that contains the left semi-derivations which satisfying some identities are commutative rings. Furthermore, the definition of semi-derivation of near-ring has been presented by Boua et. al [6]. In particular, the authors proved that the prime near-rings that satisfying some identities with semi-derivations are commutative rings. In addition, the concept of generalized semi-derivations of prime rings with some of its properties has been provided by Filippis et. al [7]. Precisely, they discussed the structure of

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$*$-prime rings that including the notion of semi-derivation which satisfying some properties on $*$-Jordan ideals of given prime ring. Moreover, the idea of generalized (σ,τ)-semi-derivations on prime near-rings has been presented by Reddy and Bharathi [8]. The authors proved some results that concern on the commutativity of prime near-ring that contains generalized (σ,τ)-semi-derivations. Some recent results which contains the idea of generalized derivations of semi-ring have been presented in some articles. For example, generalized (α,β)-derivation of prime semi-ring has been introduced by Rasheed et. al [10]. The authors used this concept to present some results that concentrated on the commutatively of the proposed structure. The commutatively of prime and semi-prime invers semi-ring has been investigated by using the concept of generalized reverse derivation [11]. In addition, the commutatively of quotient semi-ring has been proved by Mahmood et. al [12] in view of generalized derivation. While, the concept of generalized commuting mapping of prime and semi-prime rings has been presented by Mahmood [13]. In this paper, we generalized the results of [4] to the finite number of semi-rings. In other words, we introduced the generalized projective product of n-number of semi-rings. The structure of this paper is given as follows. In section two, some of the past results that are needed in this study have been stated. Section three contains the basic results of this paper and the conclusions have been provided in section four.

2. BASIC CONCEPTS

In this section, we presented some basic concepts that are needed in this study. We commence with the following definition.

Definition 2.1 [1] A self-map $\pi:\mathcal{R}\longrightarrow\mathcal{R}$ is said to be derivation on the ring $\mathcal{R}$, if $\pi(\kappa\eta)=\pi(\kappa)\eta+\kappa\pi(\eta)$ for any $\kappa,\eta\in\mathcal{R}$.

Definition 2.2 [1] A self-map $\theta:\mathcal{R}\longrightarrow\mathcal{R}$ is said to be generalized derivation (G-d) on the ring $\mathcal{R}$, if there exist a derivation $\pi:\mathcal{R}\longrightarrow\mathcal{R}$ such that $\theta(\kappa\eta)=\theta(\kappa)\eta+\kappa\pi(\eta)$ for any $\kappa,\eta\in\mathcal{R}$.

Definition 2.3 [1] A self-map $\pi:\mathcal{R}\longrightarrow\mathcal{R}$ is said to be semi-derivation (S-d) on the ring $\mathcal{R}$ associated with the function λ, if $\pi(\kappa\eta)=\pi(\kappa)\eta+\lambda(\kappa)\pi(\eta)=\pi(\kappa)\lambda(\eta)+\kappa\pi(\eta)$ and $\pi\big(\lambda(\kappa)\big)=\lambda(\pi(\kappa))$ for any $\kappa,\eta\in\mathcal{R}$.

Definition 2.4 [1] Let π be a semi-derivation of $\mathcal{R}$ associated with the function λ.Then, $\theta:\mathcal{R}\longrightarrow\mathcal{R}$ is said to be generalized semi-derivation (G-S-d) of $\mathcal{R}$, if $\theta(\kappa\eta)=\theta(\kappa)\eta+\lambda(\kappa)\pi(\eta)=\theta(\kappa)\lambda(\eta)+\kappa\pi(\eta)$ and $\theta\big(\lambda(\kappa)\big)=\lambda(\theta(\kappa))$ for any $\kappa,\eta\in\mathcal{R}$.

Definition 2.5 [2] A self-map $\pi:\mathcal{R}\longrightarrow\mathcal{R}$ is said to be Jordan derivation (J-d) on $\mathcal{R}$, if $\pi(\kappa^2)=\pi(\kappa)\kappa+\kappa\pi(\kappa)$ for any $\kappa\in\mathcal{R}$.

Definition 2.6 [3] A self-map $\theta:\mathcal{R}\longrightarrow\mathcal{R}$ is called Generalized Jordan derivation (G-J-d) of $\mathcal{R}$, if there exist a Jordan derivation $\pi:\mathcal{R}\longrightarrow\mathcal{R}$ such that $\theta(\kappa^2)=\theta(\kappa)\kappa+\kappa\pi(\kappa)$ for any $\kappa\in\mathcal{R}$.

Definition 2.7 [4] A self-map $\pi:\mathcal{R}\longrightarrow\mathcal{R}$ is said to be Jordan semi-derivation (J-S-d) on $\mathcal{R}$ associated with the function λ, if $\pi(\kappa^2)=\pi(\kappa)\lambda(\kappa)+\kappa\pi(\kappa)=\pi(\kappa)\kappa+\lambda(\kappa)\pi(\kappa)$ and $\pi\big(\lambda(\kappa)\big)=\lambda(\pi(\kappa))$ for any $\kappa\in\mathcal{R}$.

Definition 2.8 [4] A self-map $\theta:\mathcal{R}\longrightarrow\mathcal{R}$ is said to be Generalized Jordan semi-derivation (G-J-S-d) on $\mathcal{R}$ associated with Jordan semi-derivation π, if $\theta(\kappa^2)=\theta(\kappa)\lambda(\kappa)+\kappa\pi(\kappa)=\theta(\kappa)\kappa+\lambda(\kappa)\pi(\kappa)$ and $\theta\big(\lambda(\kappa)\big)=\lambda(\theta(\kappa))$ for any $\kappa\in\mathcal{R}$.

Definition 2.9 [4] Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two semi-rings and $\mathcal{R}=\mathcal{R}_1\times\mathcal{R}_2$. Two binary operations which are addition and multiplication defined on $\mathcal{R}$ by $(\kappa_1,\kappa_2)+(\eta_1,\eta_2)=(\kappa_1+\eta_1,\kappa_2+\eta_2)$ and $(\kappa_1,\kappa_2)\cdot(\eta_1,\eta_2)=(\kappa_1\cdot\eta_1,\kappa_2\cdot\eta_2)$ for any $\kappa=(\kappa_1,\kappa_2),\eta=(\eta_1,\eta_2)\in\mathcal{R}$ and $\kappa_1,\eta_1\in\mathcal{R}_1,\kappa_2,\eta_2\in\mathcal{R}_2$. Then, $\mathcal{R}$ is called the projective product of $\mathcal{R}_1\times\mathcal{R}_2$.

3. MAIN RESULTS

This section contains the main results of the present paper. We commence with the definition of generalized projective product which is given in the definition bellow.

Definition 3.1 Let $\mathcal{R}_1,\mathcal{R}_2,\cdots,\mathcal{R}_\tau$ be a finite number of semi-rings. We define the projective product of $\mathcal{R}_1,\mathcal{R}_2,\cdots,\mathcal{R}_\tau$ as a semi-ring $\mathcal{R}=\mathcal{R}_1\times\mathcal{R}_2\times\cdots\times\mathcal{R}_\tau$with the usual binary operations addition and multiplication on $\mathcal{R}$ such that $(\kappa_1,\kappa_2,\cdots,\kappa_\tau)+(\eta_1,\eta_2,\cdots,\eta_\tau)=(\kappa_1+\eta_1,\kappa_2+\eta_2,\cdots,\kappa_\tau+\eta_\tau)$ and $(\kappa_1,\kappa_2,\cdots,\kappa_\tau)\cdot(\eta_1,\eta_2,\cdots,\eta_\tau)=(\kappa_1\cdot\eta_1,\kappa_2\cdot\eta_2,\cdots,\kappa_\tau\cdot\eta_\tau)$ for any $\kappa=(\kappa_1,\kappa_2,\cdots,\kappa_\tau),\eta=(\eta_1,\eta_2,\cdots,\eta_\tau)\in\mathcal{R}$ and $\kappa_i,\eta_i\in\mathcal{R}_i$ where $i=1,\cdots,\tau$.

Theorem 3.1 Let $\mathcal{R}_i$ be a finite collection of semi-rings with $\mathcal{R}$ is their projective product for any $i = 1, \cdots, \tau$. If π_i are derivations on $\mathcal{R}_i$ for each $i = 1, \cdots, \tau$.Then, $\pi: \mathcal{R} \longrightarrow \mathcal{R}$ which is given by $\pi(\kappa) = \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) = (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau))$ is a derivation on $\mathcal{R}$ for any $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ and $\kappa_i \in \mathcal{R}_i$.

Proof: Let $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau), \eta = (\eta_1, \eta_2, \cdots, \eta_\tau) \in \mathcal{R}$ then, by Definition 2.1, we obtained $\pi(\kappa\eta) = \pi((\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau))$

$$\begin{aligned}
&= \pi(\kappa_1 \cdot \eta_1, \kappa_2 \cdot \eta_2, \cdots, \kappa_\tau \cdot \eta_\tau) \\
&= \pi_1(\kappa_1 \cdot \eta_1), \pi_2(\kappa_2 \cdot \eta_2), \cdots, \pi_\tau(\kappa_\tau \cdot \eta_\tau) \\
&= (\pi_1(\kappa_1)\eta_1 + \kappa_1\pi_1(\eta_1)), (\pi_2(\kappa_2)\eta_2 + \kappa_2\pi_2(\eta_2)), \cdots, (\pi_\tau(\kappa_\tau)\eta_\tau + \\
&\quad \kappa_\tau\pi_\tau(\eta_\tau)) \\
&= (\pi_1(\kappa_1)\eta_1, \pi_2(\kappa_2)\eta_2, \cdots, \pi_\tau(\kappa_\tau)\eta_\tau) + (\kappa_1\pi_1(\eta_1), \kappa_2\pi_2(\eta_2), \cdots \\
&\quad , \kappa_\tau\pi_\tau(\eta_\tau) \\
&= \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\eta_1, \eta_2, \cdots, \eta_\tau) \\
&= \pi(\kappa)\eta + \kappa\pi(\eta) \text{ for any } \kappa, \eta \in \mathcal{R}.
\end{aligned}$$

Therefore, π is a derivation on $\mathcal{R}$. ∎

Theorem 3.2 Let $\mathcal{R}_i$ be a finite collection of semi-rings with $\mathcal{R}$ is their projective product for any $i = 1, \cdots, \tau$. If θ_i are G-d on $\mathcal{R}_i$ for each $i = 1, \cdots, \tau$.Then, $\theta: \mathcal{R} \longrightarrow \mathcal{R}$ which is given by $\theta(\kappa) = \theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau) = (\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau))$ is a G-d on $\mathcal{R}$ for any $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ and $\kappa_i \in \mathcal{R}_i$.

Proof: Let $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau), \eta = (\eta_1, \eta_2, \cdots, \eta_\tau) \in \mathcal{R}$ then, by Definition 2.2, we have $\theta(\kappa\eta) = \theta((\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau))$

$$\begin{aligned}
&= \theta(\kappa_1 \cdot \eta_1, \kappa_2 \cdot \eta_2, \cdots, \kappa_\tau \cdot \eta_\tau) \\
&= \theta_1(\kappa_1 \cdot \eta_1), \theta_2(\kappa_2 \cdot \eta_2), \cdots, \theta_\tau(\kappa_\tau \cdot \eta_\tau) \\
&= (\theta_1(\kappa_1)\eta_1 + \kappa_1\pi_1(\eta_1)), (\theta_2(\kappa_2)\eta_2 + \kappa_2\pi_2(\eta_2)), \cdots, (\theta_\tau(\kappa_\tau)\eta_\tau + \\
&\quad \kappa_\tau\pi_\tau(\eta_\tau)) \\
&= (\theta_1(\kappa_1)\eta_1, \theta_2(\kappa_2)\eta_2, \cdots, \theta_\tau(\kappa_\tau)\eta_\tau) + (\kappa_1\pi_1(\eta_1), \kappa_2\pi_2(\eta_2), \cdots, \\
&\quad \kappa_\tau\pi_\tau(\eta_\tau)) \\
&= \theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\eta_1, \eta_2, \cdots, \eta_\tau) \\
&= \theta(\kappa)\eta + \kappa\pi(\eta) \text{ for any } \kappa, \eta \in \mathcal{R}.
\end{aligned}$$

Therefore, θ is a G-d on $\mathcal{R}$. ∎

Theorem 3.3 Let $\mathcal{R}_i$ be a finite collection of semi-rings with $\mathcal{R}$ is their projective product for any $i = 1, \cdots, \tau$. If π_i are S-d on $\mathcal{R}_i$ for each $i = 1, \cdots, \tau$.Then, $\pi: \mathcal{R} \longrightarrow \mathcal{R}$ which is given by $\pi(\kappa) = \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) = (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau))$ is a S-d on $\mathcal{R}$ for any $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ and $\kappa_i \in \mathcal{R}_i$.

Proof: Let $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau), \eta = (\eta_1, \eta_2, \cdots, \eta_\tau) \in \mathcal{R}$ then, by Definition 2.3, we get $\pi(\kappa\eta) = \pi((\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau))$

$$\begin{aligned}
&= \pi(\kappa_1 \cdot \eta_1, \kappa_2 \cdot \eta_2, \cdots, \kappa_\tau \cdot \eta_\tau) \\
&= \pi_1(\kappa_1 \cdot \eta_1), \pi_2(\kappa_2 \cdot \eta_2), \cdots, \pi_\tau(\kappa_\tau \cdot \eta_\tau) \\
&= (\pi_1(\kappa_1)\eta_1 + \lambda_1(\kappa_1)\pi_1(\eta_1)), (\pi_2(\kappa_2)\eta_2 + \lambda_2(\kappa_2)\pi_2(\eta_2)), \cdots, \\
&\quad (\pi_\tau(\kappa_\tau)\eta_\tau + \lambda_\tau(\kappa_\tau)\pi_\tau(\eta_\tau)) \\
&= (\pi_1(\kappa_1)\eta_1, \pi_2(\kappa_2)\eta_2, \cdots, \pi_\tau(\kappa_\tau)\eta_\tau) + (\lambda_1(\kappa_1)\pi_1(\eta_1), \lambda_2(\kappa_2)\pi_2(\eta_2)), \cdots, \\
&\quad \lambda_\tau(\kappa_\tau)\pi_\tau(\eta_\tau)) \\
&= (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau)) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau) + (\lambda_1(\kappa_1), \lambda_2(\kappa_2), \cdots, \\
&\quad \lambda_\tau(\kappa_\tau)) \cdot ((\pi_1(\eta_1), \pi_2(\eta_2), \cdots, \pi_\tau(\eta_\tau)) \\
&= \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau) + \lambda(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\eta_1, \eta_2, \cdots, \eta_\tau) \\
&= \pi(\kappa)\eta + \lambda(\kappa)\pi(\eta) \text{ for any } \kappa, \eta \in \mathcal{R}.
\end{aligned}$$

Also,

$$\begin{aligned}
\pi(\kappa\eta) &= \pi((\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau)) \\
&= \pi(\kappa_1 \cdot \eta_1, \kappa_2 \cdot \eta_2, \cdots, \kappa_\tau \cdot \eta_\tau) \\
&= \pi_1(\kappa_1 \cdot \eta_1), \pi_2(\kappa_2 \cdot \eta_2), \cdots, \pi_\tau(\kappa_\tau \cdot \eta_\tau) \\
&= (\pi_1(\kappa_1)\lambda_1(\eta_1) + \kappa_1\pi_1(\eta_1)), (\pi_2(\kappa_2)\lambda_2(\eta_2) + \kappa_2\pi_2(\eta_2)), \cdots, \\
&\quad (\pi_\tau(\kappa_\tau)\lambda_\tau(\eta_\tau) + \kappa_\tau\pi_\tau(\eta_\tau)) \\
&= (\pi_1(\kappa_1)\lambda_1(\eta_1), \pi_2(\kappa_2)\lambda_2(\eta_2), \cdots, \pi_\tau(\kappa_\tau)\lambda_\tau(\eta_\tau)) +
\end{aligned}$$

$((\kappa_1\pi_1(\eta_1), \kappa_2\pi_2(\eta_2), \cdots, \kappa_\tau\pi_\tau(\eta_\tau))$
$= (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau)) \cdot (\lambda_1(\eta_1), \lambda_2(\eta_2), \cdots, \lambda_\tau(\eta_\tau)) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau)$
$\cdot (\pi_1(\eta_1), \pi_2(\eta_2), \cdots, \pi_\tau(\eta_\tau))$
$= \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \lambda(\eta_1, \eta_2, \cdots, \eta_\tau) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\eta_1, \eta_2, \cdots, \eta_\tau)$
$= \pi(\kappa)\lambda(\eta) + \kappa\pi(\eta)$ for any $\kappa, \eta \in \mathcal{R}$.

Now, $\pi(\lambda(\kappa)) = \pi(\lambda(\kappa_1, \kappa_2, \cdots, \kappa_\tau))$
$= \pi(\lambda_1(\kappa_1), \lambda_2(\kappa_2), \cdots, \lambda_\tau(\kappa_\tau))$
$= \pi_1(\lambda_1(\kappa_1)), \pi_2(\lambda_2(\kappa_2)), \cdots, \pi_\tau(\lambda_\tau(\kappa_\tau))$
$= \lambda_1(\pi_1(\kappa_1)), \lambda_2(\pi_2(\kappa_2)), \cdots, \lambda_\tau(\pi_\tau(\kappa_\tau))$
$= \lambda(\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau))$
$= \lambda(\pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau))$
$= \lambda(\pi(\kappa))$.

Therefore, π is a S-d on $\mathcal{R}$. ■

Theorem 3.4 Let $\mathcal{R}_i$ be a finite collection of semi-rings with $\mathcal{R}$ is their projective product for any $i = 1, \cdots, \tau$. If θ_i are G-S-d on $\mathcal{R}_i$ for each $i = 1, \cdots, \tau$.Then, $\theta: \mathcal{R} \longrightarrow \mathcal{R}$ which is given by $\theta(\kappa) = \theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau) = (\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau))$ is a G-S-d on $\mathcal{R}$ for any $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ and $\kappa_i \in \mathcal{R}_i$.

Proof: Let $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau), \eta = (\eta_1, \eta_2, \cdots, \eta_\tau) \in \mathcal{R}$ then, by Definition 2.4, we get $\theta(\kappa\eta) = \theta((\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau))$
$= \theta(\kappa_1 \cdot \eta_1, \kappa_2 \cdot \eta_2, \cdots, \kappa_\tau \cdot \eta_\tau)$
$= \theta_1(\kappa_1 \cdot \eta_1), \theta_2(\kappa_2 \cdot \eta_2), \cdots, \theta_\tau(\kappa_\tau \cdot \eta_\tau)$
$= (\theta_1(\kappa_1)\eta_1 + \lambda_1(\kappa_1)\pi_1(\eta_1)), (\theta_2(\kappa_2)\eta_2 + \lambda_2(\kappa_2)\pi_2(\eta_2)), \cdots,$
$(\theta_\tau(\kappa_\tau)\eta_\tau + \lambda_\tau(\kappa_\tau)\pi_\tau(\eta_\tau))$
$= (\theta_1(\kappa_1)\eta_1, \theta_2(\kappa_2)\eta_2, \cdots, \theta_\tau(\kappa_\tau)\eta_\tau) + (\lambda_1(\kappa_1)\pi_1(\eta_1), \lambda_2(\kappa_2)\pi_2(\eta_2)), \cdots,$
$\lambda_\tau(\kappa_\tau)\pi_\tau(\eta_\tau)$
$= (\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau)) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau) + (\lambda_1(\kappa_1), \lambda_2(\kappa_2), \cdots,$
$\lambda_\tau(\kappa_\tau)) \cdot ((\pi_1(\eta_1), \pi_2(\eta_2), \cdots, \pi_\tau(\eta_\tau))$
$= \theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau) + \lambda(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\eta_1, \eta_2, \cdots, \eta_\tau)$
$= \theta(\kappa)\eta + \lambda(\kappa)\pi(\eta)$ for any $\kappa, \eta \in \mathcal{R}$.

Also, $\theta(\kappa\eta) = \theta((\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\eta_1, \eta_2, \cdots, \eta_\tau))$
$= \theta(\kappa_1 \cdot \eta_1, \kappa_2 \cdot \eta_2, \cdots, \kappa_\tau \cdot \eta_\tau)$
$= \theta_1(\kappa_1 \cdot \eta_1), \theta_2(\kappa_2 \cdot \eta_2), \cdots, \theta_\tau(\kappa_\tau \cdot \eta_\tau)$
$= (\theta_1(\kappa_1)\lambda_1(\eta_1) + \kappa_1\pi_1(\eta_1)), (\theta_2(\kappa_2)\lambda_2(\eta_2) + \kappa_2\pi_2(\eta_2)), \cdots,$
$(\theta_\tau(\kappa_\tau)\lambda_\tau(\eta_\tau) + \kappa_\tau\pi_\tau(\eta_\tau))$
$= (\theta_1(\kappa_1)\lambda_1(\eta_1), \theta_2(\kappa_2)\lambda_2(\eta_2), \cdots, \theta_\tau(\kappa_\tau)\lambda_\tau(\eta_\tau)) +$
$((\kappa_1\pi_1(\eta_1), \kappa_2\pi_2(\eta_2), \cdots, \kappa_\tau\pi_\tau(\eta_\tau))$
$= (\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau)) \cdot (\lambda_1(\eta_1), \lambda_2(\eta_2), \cdots, \lambda_\tau(\eta_\tau)) +$
$(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\pi_1(\eta_1), \pi_2(\eta_2), \cdots, \pi_\tau(\eta_\tau))$
$= \theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \lambda(\eta_1, \eta_2, \cdots, \eta_\tau) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\eta_1, \eta_2, \cdots, \eta_\tau)$
$= \theta(\kappa)\lambda(\eta) + \kappa\pi(\eta)$ for any $\kappa, \eta \in \mathcal{R}$.

Now, $\theta(\lambda(\kappa)) = \theta(\lambda(\kappa_1, \kappa_2, \cdots, \kappa_\tau))$
$= \theta(\lambda_1(\kappa_1), \lambda_2(\kappa_2), \cdots, \lambda_\tau(\kappa_\tau))$
$= \theta_1(\lambda_1(\kappa_1)), \theta_2(\lambda_2(\kappa_2)), \cdots, \theta_\tau(\lambda_\tau(\kappa_\tau))$
$= \lambda_1(\theta_1(\kappa_1)), \lambda_2(\theta_2(\kappa_2)), \cdots, \lambda_\tau(\theta_\tau(\kappa_\tau))$
$= \lambda(\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau))$
$= \lambda(\theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau))$
$= \lambda(\theta(\kappa))$.

Therefore, θ is a G-S-d on $\mathcal{R}$. ■

Theorem 3.5 Let $\mathcal{R}_i$ be a finite collection of semi-rings with $\mathcal{R}$ is their projective product for any $i = 1, \cdots, \tau$. If π_i are J-d on $\mathcal{R}_i$ for each $i = 1, \cdots, \tau$.Then, $\pi: \mathcal{R} \longrightarrow \mathcal{R}$ which is given by $\pi(\kappa) = \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) = (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau))$ is a J-d on $\mathcal{R}$ for any $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ and $\kappa_i \in \mathcal{R}_i$.

Proof: Let $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ then, by Definition 2.5, we obtained

$$\begin{aligned}
\pi(\kappa^2) &= \pi(\kappa_1^2, \kappa_2^2, \cdots, \kappa_\tau^2) \\
&= \pi_1(\kappa_1^2), \pi_2(\kappa_2^2), \cdots, \pi_\tau(\kappa_\tau^2) \\
&= (\pi_1(\kappa_1)\kappa_1 + \kappa_1\pi_1(\kappa_1)), (\pi_2(\kappa_2)\kappa_2 + \kappa_2\pi_2(\kappa_2)), \cdots, (\pi_\tau(\kappa_\tau)\kappa_\tau + \\
&\quad \kappa_\tau\pi_\tau(\kappa_\tau)) \\
&= (\pi_1(\kappa_1)\kappa_1, \pi_2(\kappa_2)\kappa_2, \cdots, \pi_\tau(\kappa_\tau)\kappa_\tau) + (\kappa_1\pi_1(\kappa_1), \kappa_2\pi_2(\kappa_2), \cdots, \kappa_\tau\pi_\tau(\kappa_\tau)) \\
&= (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau)) \cdot (\kappa_1, \kappa_2, \cdots, \kappa_\tau) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \\
&\quad (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau)) \\
&= \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\kappa_1, \kappa_2, \cdots, \kappa_\tau) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \\
&= \pi(\kappa)\kappa + \kappa\pi(\kappa) \text{ for any } \kappa \in \mathcal{R}.
\end{aligned}$$

Therefore, π is a J-d on $\mathcal{R}$. ■

Theorem 3.6 Let $\mathcal{R}_i$ be a finite collection of semi-rings with $\mathcal{R}$ is their projective product for any $i = 1, \cdots, \tau$. If θ_i are G-J-d on $\mathcal{R}_i$ for each $i = 1, \cdots, \tau$.Then, $\theta: \mathcal{R} \longrightarrow \mathcal{R}$ which is given by $\theta(\kappa) = \theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau) = (\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau))$ is a G-J-d on $\mathcal{R}$ for any $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ and $\kappa_i \in \mathcal{R}_i$.

Proof: Let $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ then, by Definition 2.6, we obtained

$$\begin{aligned}
\theta(\kappa^2) &= \theta(\kappa_1^2, \kappa_2^2, \cdots, \kappa_\tau^2) \\
&= \theta_1(\kappa_1^2), \theta_2(\kappa_2^2), \cdots, \theta_\tau(\kappa_\tau^2) \\
&= (\theta_1(\kappa_1)\kappa_1 + \kappa_1\pi_1(\kappa_1)), (\theta_2(\kappa_2)\kappa_2 + \kappa_2\pi_2(\kappa_2)), \cdots, (\theta_\tau(\kappa_\tau)\kappa_\tau + \kappa_\tau\pi_\tau(\kappa_\tau)) \\
&= (\theta_1(\kappa_1)\kappa_1, \theta_2(\kappa_2)\kappa_2, \cdots, \theta_\tau(\kappa_\tau)\kappa_\tau) + (\kappa_1\pi_1(\kappa_1), \kappa_2\pi_2(\kappa_2), \cdots, \kappa_\tau\pi_\tau(\kappa_\tau)) \\
&= (\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau)) \cdot (\kappa_1, \kappa_2, \cdots, \kappa_\tau) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \\
&\quad (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau)) \\
&= \theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\kappa_1, \kappa_2, \cdots, \kappa_\tau) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \\
&= \theta(\kappa)\kappa + \kappa\pi(\kappa) \text{ for any } \kappa \in \mathcal{R}.
\end{aligned}$$

Therefore, θ is a G-J-d on $\mathcal{R}$. ■

Theorem 3.7 Let $\mathcal{R}_i$ be a finite collection of semi-rings with $\mathcal{R}$ is their projective product for any $i = 1, \cdots, \tau$. If π_i are J-S-d on $\mathcal{R}_i$ for each $i = 1, \cdots, \tau$.Then, $\pi: \mathcal{R} \longrightarrow \mathcal{R}$ which is given by $\pi(\kappa) = \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) = (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau))$ is a J-S-d on $\mathcal{R}$ for any $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ and $\kappa_i \in \mathcal{R}_i$.

Proof: Let $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ then, by Definition 2.7, we have

$$\begin{aligned}
\pi(\kappa^2) &= \pi(\kappa_1^2, \kappa_2^2, \cdots, \kappa_\tau^2) \\
&= \pi_1(\kappa_1^2), \pi_2(\kappa_2^2), \cdots, \pi_\tau(\kappa_\tau^2) \\
&= (\pi_1(\kappa_1)\lambda_1(\kappa_1) + \kappa_1\pi_1(\kappa_1)), (\pi_2(\kappa_2)\lambda_2(\kappa_2) + \kappa_2\pi_2(\kappa_2)), \cdots, \\
&\quad (\pi_\tau(\kappa_\tau)\lambda_\tau(\kappa_\tau) + \kappa_\tau\pi_\tau(\kappa_\tau)) \\
&= (\pi_1(\kappa_1)\lambda_1(\kappa_1), \pi_2(\kappa_2)\lambda_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau)\lambda_\tau(\kappa_\tau)) + \\
&\quad ((\kappa_1\pi_1(\kappa_1), \kappa_2\pi_2(\kappa_2), \cdots, \kappa_\tau\pi_\tau(\kappa_\tau)) \\
&= (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau)) \cdot \big(\lambda_1(\kappa_1), \lambda_2(\kappa_2), \cdots, \lambda_\tau(\kappa_\tau)\big) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \\
&\quad \cdot (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau)) \\
&= \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \lambda(\kappa_1, \kappa_2, \cdots, \kappa_\tau) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \\
&= \pi(\kappa)\lambda(\kappa) + \kappa\pi(\kappa) \text{ for any } \kappa \in \mathcal{R}.
\end{aligned}$$

Also,

$$\begin{aligned}
\pi(\kappa^2) &= \pi(\kappa_1^2, \kappa_2^2, \cdots, \kappa_\tau^2) \\
&= \pi_1(\kappa_1^2), \pi_2(\kappa_2^2), \cdots, \pi_\tau(\kappa_\tau^2) \\
&= (\pi_1(\kappa_1)\kappa_1 + \lambda_1(\kappa_1)\pi_1(\kappa_1)), (\pi_2(\kappa_2)\kappa_2 + \lambda_2(\kappa_2)\pi_2(\kappa_2)), \cdots, (\pi_\tau(\kappa_\tau)\kappa_\tau + \\
&\quad \lambda_\tau(\kappa_\tau)\pi_\tau(\kappa_\tau) \\
&= (\pi_1(\kappa_1)\kappa_1, \pi_2(\kappa_2)\kappa_2, \cdots, \pi_\tau(\kappa_\tau)\kappa_\tau) + (\lambda_1(\kappa_1)\pi_1(\kappa_1), \lambda_2(\kappa_2)\pi_2(\kappa_2), \cdots, \\
&\quad \lambda_\tau(\kappa_\tau)\pi_\tau(\kappa_\tau) \\
&= (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau)) \cdot (\kappa_1, \kappa_2, \cdots, \kappa_\tau) + \big(\lambda_1(\kappa_1), \lambda_2(\kappa_2), \cdots, \lambda_\tau(\kappa_\tau)\big) \cdot \\
&\quad (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau))
\end{aligned}$$

$= \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\kappa_1, \kappa_2, \cdots, \kappa_\tau) + \lambda(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau)$
$= \pi(\kappa)\kappa + \lambda(\kappa)\pi(\kappa)$ for any $\kappa \in \mathcal{R}$.

Now, $\pi(\lambda(\kappa)) = \pi(\lambda(\kappa_1, \kappa_2, \cdots, \kappa_\tau))$
$= \pi(\lambda_1(\kappa_1), \lambda_2(\kappa_2), \cdots, \lambda_\tau(\kappa_\tau))$
$= \pi_1(\lambda_1(\kappa_1)), \pi_2(\lambda_2(\kappa_2)), \cdots, \pi_\tau(\lambda_\tau(\kappa_\tau))$
$= \lambda_1(\pi_1(\kappa_1)), \lambda_2(\pi_2(\kappa_2)), \cdots, \lambda_\tau(\pi_\tau(\kappa_\tau))$
$= \lambda(\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau))$
$= \lambda(\pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau))$
$= \lambda(\pi(\kappa))$.

Therefore, π is a J-S-d on $\mathcal{R}$. ■

Theorem 3.8 Let $\mathcal{R}_i$ be a finite collection of semi-rings with $\mathcal{R}$ is their projective product for any $i = 1, \cdots, \tau$. If θ_i are G-J-S-d on $\mathcal{R}_i$ for each $i = 1, \cdots, \tau$.Then, $\theta: \mathcal{R} \longrightarrow \mathcal{R}$ which is given by $\theta(\kappa) = \theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau) = (\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau))$ is a G-J-S-d on $\mathcal{R}$ for any $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ and $\kappa_i \in \mathcal{R}_i$.

Proof: Let $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \in \mathcal{R}$ then, by Definition 2.8, we have

$\theta(\kappa^2) = \theta(\kappa_1^2, \kappa_2^2, \cdots, \kappa_\tau^2)$
$= \theta_1(\kappa_1^2), \theta_2(\kappa_2^2), \cdots, \theta_\tau(\kappa_\tau^2)$
$= (\theta_1(\kappa_1)\lambda_1(\kappa_1) + \kappa_1\pi_1(\kappa_1)), (\theta_2(\kappa_2)\lambda_2(\kappa_2) + \kappa_2\pi_2(\kappa_2)), \cdots,$
$(\theta_\tau(\kappa_\tau)\lambda_\tau(\kappa_\tau) + \kappa_\tau\pi_\tau(\kappa_\tau)$
$= (\theta_1(\kappa_1)\lambda_1(\kappa_1), \theta_2(\kappa_2)\lambda_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau)\lambda_\tau(\kappa_\tau)) +$
$((\kappa_1\pi_1(\kappa_1), \kappa_2\pi_2(\kappa_2), \cdots, \kappa_\tau\pi_\tau(\kappa_\tau))$
$= (\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau)) \cdot (\lambda_1(\kappa_1), \lambda_2(\kappa_2), \cdots, \lambda_\tau(\kappa_\tau)) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot$
$(\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau))$
$= \theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \lambda(\kappa_1, \kappa_2, \cdots, \kappa_\tau) + (\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau)$
$= \theta(\kappa)\lambda(\kappa) + \kappa\pi(\kappa)$ for any $\kappa \in \mathcal{R}$.

Also,

$\theta(\kappa^2) = \theta(\kappa_1^2, \kappa_2^2, \cdots, \kappa_\tau^2)$
$= \theta_1(\kappa_1^2), \theta_2(\kappa_2^2), \cdots, \theta_\tau(\kappa_\tau^2)$
$= (\theta_1(\kappa_1)\kappa_1 + \lambda_1(\kappa_1)\pi_1(\kappa_1)), (\theta_2(\kappa_2)\kappa_2 + \lambda_2(\kappa_2)\pi_2(\kappa_2)), \cdots, (\theta_\tau(\kappa_\tau)\kappa_\tau +$
$\lambda_\tau(\kappa_\tau)\pi_\tau(\kappa_\tau)$
$= (\theta_1(\kappa_1)\kappa_1, \theta_2(\kappa_2)\kappa_2, \cdots, \theta_\tau(\kappa_\tau)\kappa_\tau) + (\lambda_1(\kappa_1)\pi_1(\kappa_1), \lambda_2(\kappa_2)\pi_2(\kappa_2), \cdots,$
$\lambda_\tau(\kappa_\tau)\pi_\tau(\kappa_\tau)$
$= (\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau)) \cdot (\kappa_1, \kappa_2, \cdots, \kappa_\tau) + (\lambda_1(\kappa_1), \lambda_2(\kappa_2), \cdots, \lambda_\tau(\kappa_\tau)) \cdot$
$(\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau))$
$= \theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot (\kappa_1, \kappa_2, \cdots, \kappa_\tau) + \lambda(\kappa_1, \kappa_2, \cdots, \kappa_\tau) \cdot \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau)$
$= \theta(\kappa)\kappa + \lambda(\kappa)\pi(\kappa)$ for any $\kappa \in \mathcal{R}$.

Now, $\theta(\lambda(\kappa)) = \theta(\lambda(\kappa_1, \kappa_2, \cdots, \kappa_\tau))$
$= \theta(\lambda_1(\kappa_1), \lambda_2(\kappa_2), \cdots, \lambda_\tau(\kappa_\tau))$
$= \theta_1(\lambda_1(\kappa_1)), \theta_2(\lambda_2(\kappa_2)), \cdots, \theta_\tau(\lambda_\tau(\kappa_\tau))$
$= \lambda_1(\theta_1(\kappa_1)), \lambda_2(\theta_2(\kappa_2)), \cdots, \lambda_\tau(\theta_\tau(\kappa_\tau))$
$= \lambda(\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau))$
$= \lambda(\theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau))$
$= \lambda(\theta(\kappa))$.

Therefore, θ is a G-J-S-d on $\mathcal{R}$. ■

Theorem 3.9 Let π be a derivation on the projective product $\mathcal{R}$ of $\prod_{i=1}^{\tau} \mathcal{R}_i$. Then, there exist a corresponding derivations π_i for any $i = 1, \cdots, \tau$ on the semi-rings $\mathcal{R}_i$ respectively.

Proof: Let π be a derivation on the projective product $\mathcal{R}$ of $\prod_{i=1}^{\tau} \mathcal{R}_i$ defined as $\pi(\kappa) = \pi(\kappa_1, \kappa_2, \cdots, \kappa_\tau) = (\pi_1(\kappa_1), \pi_2(\kappa_2), \cdots, \pi_\tau(\kappa_\tau)) = (r_1, r_2, \cdots, r_\tau)$ with $\pi_i: \mathcal{R}_i \longrightarrow \mathcal{R}_i$ are defined by $\pi_i(\kappa_i) = r_i = \chi_i(\pi(\kappa_1, \kappa_2))$ for any $\kappa_i \in \mathcal{R}_i$. To show that $\pi_1: \mathcal{R}_1 \longrightarrow \mathcal{R}_1$ is a derivation. Let $\kappa_1, \eta_1 \in \mathcal{R}_1$ then $\pi_1(\kappa_1 + \eta_1) = \chi_1(\pi(\kappa_1 + \eta_1, \kappa_2 + \eta_2)) = \chi_1(\pi(\kappa_1, \kappa_2)) + \chi_1(\pi(\eta_1, \eta_2)) = \pi_1(\kappa_1) + \pi_1(\eta_1)$. Thus, π_1 is an additive map. Now,

$\pi_1(\kappa_1\eta_1) = \chi_1(\pi(\kappa\eta))$, $\forall \kappa, \eta \in \mathcal{R}$
$= \chi_1(\pi(\kappa)\eta + \kappa\pi(\eta))$
$= \chi_1((\pi_1(\kappa_1), \pi_2(\kappa_2))(\eta_1, \eta_2) + (\kappa_1, \kappa_2)(\pi_1(\eta_1), \pi_2(\eta_2)))$
$= \chi_1((\pi_1(\kappa_1)\eta_1), (\pi_2(\kappa_2))\eta_2) + \chi_1((\kappa_1\pi_1(\eta_1)), (\kappa_2\pi_2(\eta_2)))$
$= \pi_1(\kappa_1)\eta_1 + \kappa_1\pi_1(\eta_1)$
Therefore, π_1 is a derivation on $\mathcal{R}_1$. ■

Theorem 3.10 Let θ be a G-d on the projective product $\mathcal{R}$ of $\prod_{i=1}^{\tau} \mathcal{R}_i$. Then, there exist a corresponding G-d θ_i for any $i = 1, \cdots, \tau$ on the semi-rings $\mathcal{R}_i$ respectively.
Proof: Let θ be a G-d on the projective product $\mathcal{R}$ of $\prod_{i=1}^{\tau} \mathcal{R}_i$ defined as $\theta(\kappa) = \theta(\kappa_1, \kappa_2, \cdots, \kappa_\tau) = (\theta_1(\kappa_1), \theta_2(\kappa_2), \cdots, \theta_\tau(\kappa_\tau)) = (r_1, r_2, \cdots, r_\tau)$ with $\theta_i: \mathcal{R}_i \longrightarrow \mathcal{R}_i$ are defined by $\theta_i(\kappa_i) = r_i = \chi_i(\theta(\kappa_1, \kappa_2))$ for any $\kappa_i \in \mathcal{R}_i$. To show that $\theta_1: \mathcal{R}_1 \longrightarrow \mathcal{R}_1$ is a G-d. Let $\kappa_1, \eta_1 \in \mathcal{R}_1$ then
$\theta_1(\kappa_1\eta_1) = \chi_1(\theta(\kappa\eta))$, $\forall \kappa, \eta \in \mathcal{R}$
$= \chi_1(\theta(\kappa)\eta + \kappa\pi(\eta))$
$= \chi_1((\theta_1(\kappa_1), \theta_2(\kappa_2))(\eta_1, \eta_2) + (\kappa_1, \kappa_2)(\pi_1(\eta_1), \pi_2(\eta_2)))$
$= \chi_1((\theta_1(\kappa_1)\eta_1), (\theta_2(\kappa_2))\eta_2) + \chi_1((\kappa_1\pi_1(\eta_1)), (\kappa_2\pi_2(\eta_2)))$
$= \theta_1(\kappa_1)\eta_1 + \kappa_1\pi_1(\eta_1)$
Therefore, θ_1 is a G-d on $\mathcal{R}_1$. ■

By the same way we can show that the same results for S-d/ G-S-d/ J-d/ G-J-d/ J-S-d and G-J-S-d.

4. CONCLUSION

As a result of this study, the generalization of the projective product for n-number of semi-rings which concentrated on derivations, generalized derivations, semi-derivations, generalized semi-derivations, Jordan derivations, generalized Jordan derivations, Jordan semi-derivations and generalized Jordan semi-derivations has been introduced. Moreover, the converse of these results has been also studied.

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