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Direct Product and Homomorphism of Flower

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ABSTRACT: In this paper an algebraic structure namely flower has been considered. This paper presents the notation of the direct product of two flowers and studied some of its basic properties. Then, this notation has been generalized to a finite family of flowers. Furthermore, the notation of flower homomorphism has been also studied with some of its properties. We proved some properties in view of these notations.

Keywords: Flower, Homomorphism, Direct product, Group theory, Algebraic structures.

1. INTRODUCTION

Many studies have been considered the notation of the direct product and homomorphism of several algebraic structures. For instance, the direct product of an algebraic structure namely Bitonic algebras has been introduced by Ozbal [1]. The author generalized this notation to any finite collection of Bitonic algebras. He also proved that a finite family of Bitonic algebras is commutative if and only if each one of them is commutative. Furthermore, the author presented the homomorphism of the Bitonic algebras and investigated some related results about it. Setiani et. al [2] presented the concept of the direct product of BP-algebras and studied some of its properties. The authors extended this concept to a finite family of BP-algebras. Besides that, they proved the commutativity property of the direct product of BP-algebras. Moreover, they discussed the notation of the homomorphism in view of the concept of the direct product of BP-algebras. They showed some results about this notation and generalized these results to a finite family of BP-homomorphism. In addition, the notation of the direct product has been also studied for some other algebraic structures. For example, it has been provided for B-algebras by Angeline et. al [3] and for BG-algebras by Widianto et. al [4]. The dirct product of BF-algebras has been introduced by Teves and Endam [5]. Angeline et. al [6] studied some mappings on the direct product of B-algebras. On the other hand, algebraic structures namely Flower, Garden and Farm have been introduced by Al-lahham [7]. He investigated some properties of these algebraic systems such as the commutativity property and the associativity property. Some of these algebraic systems have been extended by Atteya and Ressan [8]. They proved some results related to flower and garden such as the commutativity and some other properties. Also they determined some necessary and sufficient conditions for the structure $(T, \circledast)$ to be flower. Motivated by the works of the previous researchers, in this article we provided the concept of the direct product of Flower and the notation of the Flower homomorphism and studied some of their properties. This paper is structured as follows. In section two, some basic concepts that are needed in this research are stated. In section three, the main results of this paper are given. The conclusions of this paper and some suggestions for future works have been presented in section four.

2. BASIC CONCEPTS

This section contains some basic concepts that are needed in this research which presented as follows.

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Definition 2.1 [7] Let $T \neq \Phi$ then the binary operation $\circledast$ on T is called ATL-law if for any$x, y, z \in \mathcal{T}$ we have $X * (y * z) = z * (y * x)$.

Definition 2.2 [7] A pair $(T, \circledast)$ is called Flower if the axioms bellow are holds

1. $x * (y * z) = z * (y * x), \forall x, y, z \in \mathcal{T}$
2. $\forall x \in \mathcal{T} \exists e \in \mathcal{T}$ (right identity of T) such that $x * e = x$.
3. $x * x = e$ (right invers).

Definition 2.3 [7] Let$\Phi \neq Q \subseteq T$ then Q is sub-flower if it's a Flower with the binary operation of T.

Proposition 2.1 [7] Let$(T, \circledast)$ be a Flower. Then, $(y * z) * x = (y * x) * z$ for any $x, y, z \in \mathcal{T}$.

Proposition 2.2 [7] Let$(T, \circledast)$ be a Flower. Then, the points below are equivalent

1. $\mathcal{T}$Is Lahhmian group
2. $\mathcal{T}$ Has identity
3. $\mathcal{T}$ Is abelian
4. $\mathcal{T}$ Is an associative

Theorem 2.1 [8] Every flower is commutative.

3. MAIN RESULTS

This section deals with the notations of the direct product and homomorphism of flower. We start with the following definition.

Definition 3.1 Let $(T_1, \circledast)$ and $(T_2, \diamond)$ be two flowers. We define the direct product of T_1 and T_2 as a structure $(T_1 \times T_2, \boxtimes)$ such that all the points below are fulfilled.

1. $(x_1, x_2) \otimes ((y_1, y_2) \otimes (z_1, z_2)) = (z_1, z_2) \otimes ((y_1, y_2) \otimes (x_1, x_2))$ for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathcal{T}_1 \times \mathcal{T}_2$. .
2. For any $(X_1, X_2) \in T_1 \times T_2$ there exist $(e_1, e_2) \in T_1 \times T_2$ such that $(X_1, X_2) \boxtimes (e_1, e_2) = (X_1, X_2)$ where (e_1, e_2) is the right identity of $T_1 \times T_2$.
3. For any $(x_1, x_2) \in \mathcal{T}_1 \times \mathcal{T}_2$ we have $(x_1, x_2) \otimes (x_1, x_2) = (e_1, e_2)$

Theorem 3.1 A structures $(T_1, \circledast)$ and $(T_2, \diamond)$ are flowers iff $(T_1 \times T_2, \boxtimes)$ is a flower.

Proof: Suppose that $(T_1, \circledast)$ and $(T_2, \diamond)$ are flowers to show that $(T_1 \times T_2, \boxtimes)$ is a flower. Since$(T_1, \circledast)$ and $(T_2, \diamond)$ are flowers, then for each $X_1, y_1, z_1 \in T_1$ and $X_2, y_2, z_2 \in T_2$ we have the following points (1) $(X_1 \circledast (y_1 \circledast z_1), X_2 \diamond (y_2 \diamond z_2)) = (z_1 \circledast (y_1 \circledast X_1), z_2 \diamond (y_2 \diamond X_2))$. This gives us that $(X_1, X_2) \boxtimes ((y_1 \circledast z_1), (y_2 \diamond z_2)) = (z_1, z_2) \boxtimes ((y_1 \circledast X_1), (y_2 \diamond X_2))$ which implies that $(X_1, X_2) \boxtimes ((y_1, y_2) \boxtimes (z_1, z_2)) = (z_1, z_2) \boxtimes ((y_1, y_2) \boxtimes (X_1, X_2))$ for any $(X_1, X_2), (y_1, y_2), (z_1, z_2) \in T_1 \times T_2$. (2) From the assumption, for any $X_1 \in T_1$ and $X_2 \in T_2$ there exist $e_1 \in T_1$ and $e_2 \in T_2$ such that $X_1 \circledast e_1 = X_1$ and $X_2 \diamond e_2 = X_2$. That is mean $(X_1, X_2) \boxtimes (e_1, e_2) = (X_1, X_2)$ which gives that (e_1, e_2) is the right identity of $T_1 \times T_2$. (3) Again from the assumption we have $X_1 \circledast X_1 = e_1$ and $X_2 \diamond X_2 = e_2$.That is mean $(X_1, X_2) \boxtimes (X_1, X_2) = (e_1, e_2)$ for any $(X_1, X_2) \in T_1 \times T_2$. Therefore, $(T_1 \times T_2, \boxtimes)$ is a flower. Conversely, let $(T_1 \times T_2, \boxtimes)$ be a flower to prove that $(T_1, \circledast)$ and $(T_2, \diamond)$ are flowers. Since $(T_1 \times T_2, \boxtimes)$ is a flower, then we have the following points (1) $(X_1, X_2) \boxtimes ((y_1, y_2) \boxtimes (z_1, z_2)) = (z_1, z_2) \boxtimes ((y_1, y_2) \boxtimes (X_1, X_2))$ for all $(X_1, X_2), (y_1, y_2), (z_1, z_2) \in T_1 \times T_2$. This gives that $(X_1, X_2) \boxtimes ((y_1 \circledast z_1), (y_2 \diamond z_2)) = (z_1, z_2) \boxtimes ((y_1 \circledast X_1), (y_2 \diamond X_2)) \Longrightarrow (X_1 \circledast (y_1 \circledast z_1), X_2 \diamond (y_2 \diamond z_2)) = (z_1 \circledast (y_1 \circledast X_1), z_2 \diamond (y_2 \diamond X_2))$. That is mean $X_1 \circledast (y_1 \circledast z_1) = z_1 \circledast (y_1 \circledast X_1)$ and $X_2 \diamond (y_2 \diamond z_2) = z_2 \diamond (y_2 \diamond X_2)$ for all $X_1, y_1, z_1 \in T_1$ and $X_2, y_2, z_2 \in T_2$. (2) Since (e_1, e_2) is the right identity of $T_1 \times T_2$, then we have $(X_1, X_2) \boxtimes (e_1, e_2) = (X_1, X_2)$ implies $(X_1 \circledast e_1, X_2 \diamond e_2) = (X_1, X_2)$ which gives $X_1 \circledast e_1 = X_1$ and $X_2 \diamond e_2 = X_2$ for any $X_1 \in T_1$ and $X_2 \in T_2$. (3) Also from the assumption we have that $(X_1, X_2) \boxtimes (X_1, X_2) = (e_1, e_2)$ for any $(X_1, X_2) \in T_1 \times T_2$. That is mean $(X_1 \circledast X_1, X_2 \diamond X_2) = (e_1, e_2)$ which gives that $X_1 \circledast X_1 = e_1$ and $X_2 \diamond X_2 = e_2$ for any $X_1 \in T_1$ and $X_2 \in T_2$. Therefore, $(T_1, \circledast)$ and $(T_2, \diamond)$ are flowers.

Corollary 3.1 The direct product of finite family of flowers is a flower.

Theorem 3.2 A system $(T_1 \times T_2, \boxtimes)$ is an abelian iff $(T_1, \circledast)$ and $(T_2, \diamond)$ are abelian.

Proof: Let $(T_1 \times T_2, \boxtimes)$ is an abelian then for any $(X_1, X_2), (y_1, y_2) \in T_1 \times T_2$ we have that $(X_1, X_2) \boxtimes (y_1, y_2) = (y_1, y_2) \boxtimes (X_1, X_2) \Longrightarrow (X_1 \circledast y_1, X_2 \diamond y_2) = (y_1 \circledast X_1, y_2 \diamond X_2)$. This gives that $X_1 \circledast y_1 = y_1 \circledast X_1$ and $X_2 \diamond y_2 = y_2 \diamond X_2$ which proved that $(T_1, \circledast)$ and $(T_2, \diamond)$ are abelian. Conversely, let $(T_1, \circledast)$ and $(T_2, \diamond)$ are abelian, then for any $X_1, y_1 \in T_1$ and $X_2, y_2 \in T_2$ we have $X_1 \circledast y_1 = y_1 \circledast X_1$ and $X_2 \diamond y_2 = y_2 \diamond X_2$. This implies that $(X_1 \circledast y_1, X_2 \diamond y_2) = (y_1 \circledast X_1, y_2 \diamond X_2) \Longrightarrow (X_1, X_2) \boxtimes (y_1, y_2) = (y_1, y_2) \boxtimes (X_1, X_2)$ which proved $(T_1 \times T_2, \boxtimes)$ is an abelian.

Theorem 3.3 A finite intersection of flowers is a flower.

Proof: Let $\{(T_i, \circledast) : i = 1, \cdots, v\}$ be a finite family of flowers to show that $\{(\cap T_i, \circledast) : i = 1, \cdots, v\}$ is form a flower. Let $X_i, y_i, z_i \in \bigcap_{i=1}^{v} (T_i, \circledast)$, then $X_i, y_i, z_i \in (T_i, \circledast)$ for all $i = 1, \cdots, v$. Since $(T_i, \circledast)$ is a flower for any $i = 1, \cdots, v$, then we have $X_i \circledast (y_i \circledast z_i) = z_i \circledast (y_i \circledast X_i)$ for any $i = 1, \cdots, v$ which gives that condition one is hold. Since $X_i \in \bigcap_{i=1}^{v} (T_i, \circledast)$ then $X_i \in (T_i, \circledast)$ for all $i = 1, \cdots, v$. Since $(T_i, \circledast)$ is a flower for any $i = 1, \cdots, v$, there exist a right identities $e_i \in (T_i, \circledast)$ for all $i = 1, \cdots, v$ such that $X_i \circledast e_i = X_i$ for any $i = 1, \cdots, v$. Thus, $\bigcap_{i=1}^{v} (T_i, \circledast)$ has a right identity. Finally, we have

$X_i \in \bigcap_{i=1}^{v}(T_i, \circledast)$ which implies $X_i \in (T_i, \circledast)$ for all $i = 1, \cdots, v$. Since $(T_i, \circledast)$ is a flower for any $i = 1, \cdots, v$, then we have $X_i \circledast X_i = e_i$ for any $i = 1, \cdots, v$. Thus, condition three is proved. Therefore, $\{(\cap T_i, \circledast) : i = 1, \cdots, v\}$ is a flower.

Corollary 3.2 The finite intersection of a finite direct product of flowers is a flower.

Proof: Let $\{(T_i, \circledast) : i = 1, \cdots, v\}$ be a finite family of flowers. By Corollary 3.1, $(\prod_{i=1}^{v} T_i, \circledast)$ is a flower. By Theorem 3.3, $\left(\bigcap_{i=1}^{v} T_i, \circledast\right)$ is a flower. Thus, $(\cap \prod_{i=1}^{v} T_i, \circledast) = (\prod_{i=1}^{v} \cap T_i, \circledast)$ which is flower. Therefore, as required.

Theorem 3.4 Let $(T_1 \times T_2, \boxtimes)$ be a flower. Then, for each $(X_1, X_2), (y_1, y_2) \in T_1 \times T_2$, the following points are holds.

1. $(X_1, X_2) \boxtimes ((X_1, X_2) \boxtimes (y_1, y_2)) = (y_1, y_2)$.
2. $(y_1, y_2) \boxtimes (X_1, X_2) = (e_1, e_2) \boxtimes ((X_1, X_2) \boxtimes (y_1, y_2))$.
3. $(e_1, e_2) \boxtimes ((X_1, X_2) \boxtimes (y_1, y_2)) = ((e_1, e_2) \boxtimes (X_1, X_2)) \boxtimes ((e_1, e_2) \boxtimes (y_1, y_2))$.
4. $(X_1, X_2) \boxtimes (y_1, y_2) = (X_1, X_2) \Longleftrightarrow (y_1, y_2) = (e_1, e_2)$.
5. $(X_1, X_2) \boxtimes (y_1, y_2) = (e_1, e_2) \Longleftrightarrow (y_1, y_2) = (X_1, X_2)$.
6. $((X_1, X_2) \boxtimes (y_1, y_2)) \boxtimes (X_1, X_2) = (e_1, e_2) \boxtimes (y_1, y_2)$.
7. $((X_1, X_2) \boxtimes (e_1, e_2)) \boxtimes (X_1, X_2) = ((X_1, X_2) \boxtimes (X_1, X_2)) = (e_1, e_2)$.
8. $(X_1, X_2) \boxtimes ((X_1, X_2) \boxtimes (e_1, e_2)) = ((X_1, X_2) \boxtimes (X_1, X_2)) = (e_1, e_2)$.

Proof: (1) $(X_1, X_2) \boxtimes ((X_1, X_2) \boxtimes (y_1, y_2)) = (X_1, X_2) \boxtimes ((X_1 \circledast y_1, X_2 \diamond y_2)) =$
$(X_1 \circledast (X_1 \circledast y_1), X_2 \diamond (X_2 \diamond y_2))$
$= (y_1 \circledast (X_1 \circledast X_1), y_2 \diamond (X_2 \diamond X_2))$ (by ATL-law)
$= (y_1 \circledast e_1, y_2 \diamond e_2)$
$= (y_1, y_2)$.

(2) $(e_1, e_2) \otimes ((x_1, x_2) \otimes (y_1, y_2)) = (e_1, e_2) \otimes ((x_1 * y_1, x_2 \diamond y_2))$
$= (e_1 \circledast (X_1 \circledast y_1), e_2 \diamond (X_2 \diamond y_2))$
$= (y_1 \circledast (X_1 \circledast e_1), y_2 \diamond (X_2 \diamond e_2))$ (by ATL-law)
$= (y_1 \circledast X_1, y_2 \diamond X_2)$ (as e_1, e_2 are right identities)
$= (y_1, y_2) \boxtimes (X_1, X_2)$.

(3) $((e_1, e_2) \boxtimes (X_1, X_2)) \boxtimes ((e_1, e_2) \boxtimes (y_1, y_2))$
$= (e_1 \circledast X_1, e_2 \diamond X_2) \boxtimes (e_1 \circledast y_1, e_2 \diamond y_2)$
$= ((e_1 \circledast X_1) \circledast (e_1 \circledast y_1), (e_2 \diamond X_2) \diamond (e_2 \diamond y_2))$
$= (e_1 \circledast ((X_1 \circledast e_1) \circledast y_1), e_2 \diamond ((X_2 \diamond e_2) \diamond y_2))$ (by Proposition 2.2)
$= (y_1 \circledast ((X_1 \circledast e_1) \circledast e_1), y_2 \diamond ((X_2 \diamond e_2) \diamond e_2))$ (by ATL-law)
$= (y_1 \circledast (X_1 \circledast e_1), y_2 \diamond (X_2 \diamond e_2))$
$= (e_1 \circledast (X_1 \circledast y_1), e_2 \diamond (X_2 \diamond y_2))$ (by ATL-law)
$= (e_1, e_2) \boxtimes (X_1 \circledast y_1, X_2 \diamond y_2)$
$= (e_1, e_2) \boxtimes ((X_1 \circledast X_2) \boxtimes (y_1 \diamond y_2))$.

(4) Let $(x_1, x_2) \otimes (y_1, y_2) = (x_1, x_2)$
$\Longrightarrow (X_1 \circledast y_1, X_2 \diamond y_2) = (X_1, X_2)$. That is mean $X_1 \circledast y_1 = X_1$ and $X_2 \diamond y_2 = X_2$ .This gives that $y_1 = e_1$ and $y_2 = e_2$ which implies that $(y_1, y_2) = (e_1, e_2)$. Conversely, let $(y_1, y_2) = (e_1, e_2)$ then $y_1 = e_1$ and $y_2 = e_2$. Left multiply by X_1 and X_2 respectively we get $X_1 \circledast y_1 = X_1 \circledast e_1$ and $X_2 \diamond y_2 = X_2 \diamond e_2$. Since e_1 and e_2 are the right identities, then $X_1 \circledast y_1 = X_1$ and $X_2 \diamond y_2 = X_2$ which implies that $(X_1 \circledast y_1, X_2 \diamond y_2) = (X_1, X_2) \Longrightarrow (X_1, X_2) \boxtimes (y_1, y_2) = (X_1, X_2)$.

(5) Let $(X_1, X_2) \boxtimes (y_1, y_2) = (e_1, e_2)$ then $(X_1 \circledast y_1, X_2 \diamond y_2) = (e_1, e_2)$. That is mean $X_1 \circledast y_1 = e_1$ and $X_2 \diamond y_2 = e_2$. Right multiply by y_1 and y_2 respectively, we have $X_1 \circledast y_1 \circledast y_1 = e_1 \circledast y_1$ and $X_2 \diamond y_2 \diamond y_2 = e_2 \diamond y_2$which implies $X_1 = e_1 \circledast y_1$ and $X_2 = e_2 \diamond y_2$. By Theorem 2.1, then $X_1 = y_1$ and $X_2 = y_2$ which gives $(y_1, y_2) = (X_1, X_2)$. Conversely, let $(y_1, y_2) = (X_1, X_2)$ then $X_1 = y_1$ and $X_2 = y_2$. Left multiply by y_1 and y_2 respectively, we get $y_1 \circledast X_1 = y_1 \circledast y_1$ and $y_2 \diamond X_2 = y_2 \diamond y_2$. This implies that $y_1 \circledast X_1 = e_1$ and $y_2 \diamond X_2 = e_2$ which gives $(y_1 \circledast X_1, y_2 \diamond X_2) = (e_1, e_2)$. By Theorem 2.1, $(X_1, X_2) \boxtimes (y_1, y_2) = (e_1, e_2)$.

(6) $((x_1, x_2) \otimes (y_1, y_2)) \otimes (x_1, x_2) = (x_1 * y_1, x_2 \circ y_2) \otimes (x_1, x_2)$
$= (X_1 \circledast (y_1 \circledast X_1), X_2 \diamond (y_2 \diamond X_2))$
$= (X_1 \circledast X_1 \circledast y_1, X_2 \diamond X_2 \diamond y_2)$ (by Theorem 2.1)
$= (e_1 \circledast y_1, e_2 \diamond y_2)$
$= (e_1, e_2) \boxtimes (y_1, y_2)$.

(7) $((x_1, x_2) \otimes (e_1, e_2)) \otimes (x_1, x_2) = (x_1 * e_1, x_2 \diamond e_2) \otimes (x_1, x_2)$
$= (X_1, X_2) \boxtimes (X_1, X_2)$
$= (X_1 \circledast X_1, X_2 \diamond X_2)$
$= (e_1, e_2)$.

(8) $(x_1, x_2) \otimes ((x_1, x_2) \otimes (e_1, e_2)) = (x_1, x_2) \otimes (x_1 * e_1, x_2 \diamond e_2)$
$= (X_1, X_2) \boxtimes (X_1, X_2)$
$= (e_1, e_2)$.

Theorem 3.5 Let $(T_1 \times T_2,\ \boxtimes)$ be a flower. Then, for each $(X_1, X_2), (y_1, y_2), (z_1, z_2) \in T_1 \times T_2$, the following points are holds.

1. $((X_1, X_2) \boxtimes (y_1, y_2)) \boxtimes ((X_1, X_2) \boxtimes (z_1, z_2)) = (z_1, z_2) \boxtimes (y_1, y_2)$.
2. $(((X_1, X_2) \boxtimes (y_1, y_2)) \boxtimes ((X_1, X_2) \boxtimes (z_1, z_2))] \boxtimes ((z_1, z_2) \boxtimes (y_1, y_2)) =$
(e_1, e_2).
3. $((X_1, X_2) \boxtimes ((y_1, y_2) \boxtimes (X_1, X_2))] \boxtimes ((X_1, X_2) \boxtimes (y_1, y_2)) = (X_1, X_2)$.
4. $((X_1, X_2) \boxtimes (y_1, y_2)) \boxtimes (z_1, z_2) = (X_1, X_2) \boxtimes ((y_1, y_2) \boxtimes ((e_1, e_2) \boxtimes (z_1, z_2))]$.

Proof: (1) $((X_1, X_2) \boxtimes (y_1, y_2)) \boxtimes ((X_1, X_2) \boxtimes (z_1, z_2))$
$= (X_1 \divideontimes y_1, X_2 \diamond y_2) \boxtimes (X_1 \divideontimes z_1, X_2 \diamond z_2)$
$= (X_1 \divideontimes y_1) \divideontimes (X_1 \divideontimes z_1), (X_2 \diamond y_2) \diamond (X_2 \diamond z_2)$
$= X_1 \divideontimes ((y_1 \divideontimes X_1) \divideontimes z_1), X_2 \diamond ((y_2 \diamond X_2) \diamond z_2)$
$= z_1 \divideontimes ((y_1 \divideontimes X_1) \divideontimes X_1), z_2 \diamond ((y_2 \diamond X_2) \diamond X_2)$ (by ATL-law)
$= (z_1 \divideontimes y_1, z_2 \diamond y_2)$
$= (z_1, z_2) \boxtimes (y_1, y_2)$.
(2) $[((x_1, x_2) \otimes (y_1, y_2)) \otimes ((x_1, x_2) \otimes (z_1, z_2))] \otimes ((z_1, z_2) \otimes (y_1, y_2)) =$
$(X_1 \divideontimes y_1, X_2 \diamond y_2) \boxtimes (X_1 \divideontimes z_1, X_2 \diamond z_2) \boxtimes (z_1 \divideontimes y_1, z_2 \diamond y_2) =$
$(X_1 \divideontimes y_1 \divideontimes X_1 \divideontimes z_1 \divideontimes z_1 \divideontimes y_1, X_2 \diamond y_2 \diamond X_2 \diamond z_2 \diamond z_2 \diamond y_2)$
$= X_1 \divideontimes ((y_1 \divideontimes X_1) \divideontimes y_1), X_2 \diamond ((y_2 \diamond X_2) \diamond y_2)$
$= y_1 \divideontimes y_1 \divideontimes X_1 \divideontimes X_1, y_2 \diamond y_2 \diamond X_2 \diamond X_2$ (by ATL-law)
$= (e_1, e_2)$.
(3) $[(x_1, x_2) \otimes ((y_1, y_2) \otimes (x_1, x_2))] \otimes ((x_1, x_2) \otimes (y_1, y_2)) =$
$(X_1, X_2) \boxtimes (y_1 \divideontimes X_1, y_2 \diamond X_2) \boxtimes (X_1 \divideontimes y_1, X_2 \diamond y_2)$
$= (X_1 \divideontimes y_1 \divideontimes X_1 \divideontimes X_1 \divideontimes y_1, X_2 \diamond y_2 \diamond X_2 \diamond X_2 \diamond y_2)$
$= (X_1 \divideontimes y_1 \divideontimes y_1, X_2 \diamond y_2 \diamond y_2)$
$= (X_1, X_2)$.
(4) $(X_1, X_2) \boxtimes ((y_1, y_2) \boxtimes ((e_1, e_2) \boxtimes (z_1, z_2))] = (X_1, X_2) \boxtimes (y_1 \divideontimes e_1 \divideontimes z_1, y_2 \diamond e_2 \diamond z_2) = (X_1 \divideontimes y_1 \divideontimes z_1, X_2 \diamond y_2 \diamond z_2) = (X_1 \divideontimes y_1, X_2 \diamond y_2) \boxtimes (z_1, z_2) = ((X_1, X_2) \boxtimes (y_1, y_2)) \boxtimes (z_1, z_2)$

Corollary 3.3 Let $(T_1 \times T_2,\ \boxtimes)$ be a flower. Then,

1. $(T_1 \times T_2,\ \boxtimes)$ has unique right identity.
2. For each $(x_1, x_2) \in \mathcal{T}_1 \times \mathcal{T}_2$ has unique right inverse

Proof: (1) Let $(e_1, e_2), \left(e_1', e_2'\right)$ be two right identities of $T_1 \times T_2$. If $(e_1, e_2)$is the right identity then for any $(X_1, X_2) \in T_1 \times T_2$ we have $(X_1, X_2) \boxtimes (e_1, e_2) = (X_1 \divideontimes e_1, X_2 \diamond e_2) = (X_1, X_2)$. Also, if $\left(e_1', e_2'\right)$ is the right identity then we have $(X_1, X_2) \boxtimes \left(e_1', e_2'\right) = \left(X_1 \divideontimes e_1', X_2 \diamond e_2'\right) = (X_1, X_2)$. That is mean $(X_1, X_2) \boxtimes (e_1, e_2) = (X_1, X_2) \boxtimes \left(e_1', e_2'\right) \Longrightarrow (e_1, e_2) = \left(e_1', e_2'\right)$ which proved the uniqueness of the right identity. By the same way we can prove point two.

Theorem 3.6 Let $(T_1 \times T_2,\ \boxtimes)$ be a flower. Then, for each $(X_1, X_2), (y_1, y_2), (z_1, z_2) \in T_1 \times T_2$ we have $((y_1, y_2) \boxtimes (z_1, z_2)) \boxtimes (X_1, X_2) = ((y_1, y_2) \boxtimes (X_1, X_2)) \boxtimes (z_1, z_2)$.

Proof: $((y_1, y_2) \boxtimes (z_1, z_2)) \boxtimes (X_1, X_2) = (y_1 \divideontimes z_1, y_2 \diamond z_2) \boxtimes (X_1, X_2) = (y_1 \divideontimes z_1 \divideontimes X_1, y_2 \diamond z_2 \diamond X_2)$. By Theorem 2.1, we have $= (y_1 \divideontimes X_1 \divideontimes z_1, y_2 \diamond X_2 \diamond z_2) = (y_1 \divideontimes X_1, y_2 \diamond X_2) \boxtimes (z_1, z_2) = ((y_1, y_2) \boxtimes (X_1, X_2)) \boxtimes (z_1, z_2)$. Therefore, as required.

Theorem 3.7 Let $(T_1 \times T_2,\ \boxtimes)$ be a flower. If $(X_1, X_2) \boxtimes (y_1, y_2) = (z_1, z_2) \boxtimes (W_1, W_2)$ then $(y_1, y_2) = (z_1, z_2)$ for any $(X_1, X_2), (y_1, y_2), (z_1, z_2), (W_1, W_2) \in T_1 \times T_2$.

Proof: We have $(X_1, X_2) \boxtimes (y_1, y_2) = (z_1, z_2) \boxtimes (W_1, W_2)$ that is mean $(X_1 \divideontimes y_1, X_2 \diamond y_2) = (z_1 \divideontimes W_1, z_2 \diamond W_2)$. From this we get $X_1 \divideontimes y_1 = z_1 \divideontimes W_1$ and $X_2 \diamond y_2 = z_2 \diamond W_2$. We take the first equation which is $X_1 \divideontimes y_1 = z_1 \divideontimes W_1$. Left multiple by X_1 we obtain $X_1 \divideontimes (X_1 \divideontimes y_1) = X_1 \divideontimes (z_1 \divideontimes W_1)$. Using ATL-law, we have $y_1 \divideontimes (X_1 \divideontimes X_1) = W_1 \divideontimes (z_1 \divideontimes X_1) \Longrightarrow y_1 = (W_1 \divideontimes z_1) \divideontimes X_1$. By Proposition 2.1, we get $y_1 = (W_1 \divideontimes X_1) \divideontimes z_1$. Replace W_1 by X_1 we have $y_1 = X_1 \divideontimes (X_1 \divideontimes z_1)$. Using ATL-law, we have $y_1 = z_1 \divideontimes (X_1 \divideontimes X_1) \Longrightarrow y_1 = z_1$. By the same way we can prove that $y_2 = z_2$. Therefore, $(y_1, y_2) = (z_1, z_2)$.

Theorem 3.8 If $(T_1 \times T_2,\ \cdot)$ be an abelian group. Then, $(T_1 \times T_2,\ \boxdot)$ is a flower such that $(X_1, X_2) \boxdot (y_1, y_2) = (X_1, X_2) \cdot (y_1, y_2)^{-1}$ for any $(X_1, X_2), (y_1, y_2) \in T_1 \times T_2$.

Proof: (1) For any $(X_1, X_2), (y_1, y_2), (z_1, z_2) \in T_1 \times T_2$ then
$(X_1, X_2) \boxdot (y_1, y_2) \boxdot (z_1, z_2) = (X_1, X_2) \boxdot \left((y_1, y_2) \cdot (z_1, z_2)^{-1}\right)$
$= (X_1, X_2) \cdot \left((y_1, y_2) \cdot (z_1, z_2)^{-1}\right)^{-1}$
$= (X_1, X_2) \cdot \left((z_1, z_2) \cdot (y_1, y_2)^{-1}\right)$
$= (X_1, X_2) \cdot \left((z_1, z_2) \cdot (y_1^{-1}, y_2^{-1})\right)$
$= (X_1, X_2) \cdot \left(z_1 \cdot y_1^{-1}, z_2 \cdot y_2^{-1}\right)$

$= (X_1 \cdot z_1) \cdot y_1^{-1}, (X_2 \cdot z_2) \cdot y_2^{-1}$
$= (z_1 \cdot X_1) \cdot y_1^{-1}, (z_2 \cdot X_2) \cdot y_2^{-1}$(by assumption)
$= (z_1, z_2) \cdot \left(X_1 \cdot y_1^{-1}, X_2 \cdot y_2^{-1}\right)$
$= (z_1, z_2) \cdot \left((X_1, X_2) \cdot \left(y_1^{-1}, y_2^{-1}\right)\right)$
$= (z_1, z_2) \cdot \left((X_1, X_2) \cdot (y_1, y_2)^{-1}\right)$
$= (z_1, z_2) \cdot \left((y_1, y_2) \cdot (X_1, X_2)^{-1}\right)^{-1}$
$= (z_1, z_2) \boxdot \left((y_1, y_2) \cdot (X_1, X_2)^{-1}\right)$
$= (z_1, z_2) \boxdot ((y_1, y_2) \boxdot (X_1, X_2)).$

(2) For any $(X_1, X_2) \in T_1 \times T_2$ there exist $(e_1, e_2) \in T_1 \times T_2$ such that $(X_1, X_2) \boxdot (e_1, e_2) = (X_1, X_2) \cdot (e_1, e_2)^{-1} = (X_1, X_2) \cdot \left(e_1^{-1}, e_2^{-1}\right) = (X_1, X_2) \cdot (e_1, e_2) = (X_1 \cdot e_1, X_2 \cdot e_2) = (X_1, X_2)$. (3) For any $(X_1, X_2) \in T_1 \times T_2$ we have $(X_1, X_2) \boxdot (X_1, X_2) = (X_1, X_2) \cdot (X_1, X_2)^{-1} = (X_1, X_2) \cdot \left(X_1^{-1}, X_2^{-1}\right) = \left(X_1 \cdot X_1^{-1}, X_2 \cdot X_2^{-1}\right) = (e_1, e_2)$. Therefore, $(T_1 \times T_2, \ \boxdot)$ is a flower.

Definition 3.2 Let $(T_1, \divideontimes)$ and $(T_2, \diamond)$ be two flowers. We define the flowers homomorphism as a mapping $f : T_1 \longrightarrow T_2$ in which $f(X \divideontimes y) = f(X) \diamond f(y)$ for any $X, y \in T_1$.

Remark 3.1 Let $f : T_1 \longrightarrow T_2$ be a flowers homomorphism. If f is an onto, then it's called is an epimorphism, if f is (1-1) then it's called monomorphism and if f is bijection then it's called isomorphism.

Theorem 3.9 The homomorphic image of a flower is a flower.

Proof: Let $f : (T_1, \divideontimes) \longrightarrow (T_2, \diamond)$ be a flowers homomorphism to show that $f(T_1)$ is a flower. Let $X_1, y_1, z_1 \in T_1$ then $f(X_1), f(y_1), f(z_1) \in f(T_1)$. Since $X_1, y_1, z_1 \in T_1$ and $(T_1, \divideontimes)$ is a flower, then we get $X_1 \divideontimes (y_1 \divideontimes z_1) = z_1 \divideontimes (y_1 \divideontimes X_1)$. Since f be a homomorphism, then $f(X_1) \divideontimes f(y_1 \divideontimes z_1) = f(z_1) \divideontimes f(y_1 \divideontimes X_1)$ which implies $f(X_1) \divideontimes (f(y_1) \divideontimes f(z_1)) = f(z_1) \divideontimes (f(y_1) \divideontimes f(X_1)) = f(X_1) \diamond (f(y_1) \diamond f(z_1)) = f(z_1) \diamond (f(y_1) \diamond f(X_1))$. Thus, condition one is hold. Since $X_1 \in T_1$ and $(T_1, \divideontimes)$ is a flower, then there exist a right identity $e_1 \in T_1$ such that $X_1 \divideontimes e_1 = X_1$. By homomorphism of f, we have $f(X_1) \diamond f(e_1) = f(X_1) \Longrightarrow f(X_1 \divideontimes e_1) = f(X_1)$ which proves condition two. Finally, since $X_1 \in T_1$ and $(T_1, \divideontimes)$ is a flower, then we have $X_1 \divideontimes X_1 = e_1$. This gives that $f(X_1 \divideontimes X_1) = f(e_1) \Longrightarrow f(X_1) \diamond f(X_1) = f(e_1)$. So, condition three is hold. Therefore, $f(T_1)$ is a flower.

Corollary 3.4 Let $f : (T_1, \divideontimes) \longrightarrow (T_2, \diamond)$ be a flowers homomorphism. If e_1 is the right identity of T_1, then $f(e_1)$ is the right identity of T_2.

Proof: Let e_1 be a right identity of T_1, then for any $X_1 \in T_1$ we have $X_1 \divideontimes e_1 = X_1 \Longrightarrow f(X_1 \divideontimes e_1) = f(X_1)$ implies $f(X_1) \diamond f(e_1) = f(X_1)$ which gives that $f(e_1)$ is the right identity of T_2.

Theorem 3.10 Let $f : (T_1, \divideontimes) \longrightarrow (T_2, \diamond)$ be a flowers homomorphism. Then,

1. If $\mathfrak{Q}$ is a sub-flower of T_1, then f($\mathfrak{Q}$ is a sub-flower of T_2
2. If $\mathfrak{Q}$ is a sub-flower of T_2, then $f^{-1}(\mathfrak{Q})$ is a sub-flower of T_1

Proof: (1) Let Q be a sub-flower of T_1, then by Definition 2.3, we have Q is a flower. By Theorem 3.9, $f(Q)$is a flower of T_2. (2) Let $f^{-1}(Q) = \{X_1 \in T_1 : f(X_1) \in Q\}$. That is mean $f(X_1) \in Q \Longleftrightarrow X_1 \in f^{-1}(Q)$. Since Q is a sub-flower of T_2, then by Definition 2.3, Q is a flower. This gives us that if $f(X_1), f(y_1), f(z_1) \in Q$ where $X_1, y_1, z_1 \in f^{-1}(Q)$ we have that $f(X_1) \diamond (f(y_1) \diamond f(z_1)) = f(z_1) \diamond (f(y_1) \diamond f(X_1)) \Longrightarrow X_1 \divideontimes (y_1 \divideontimes z_1) = z_1 \divideontimes (y_1 \divideontimes X_1)$. Thus condition one is hold. For condition two, if $f(X_1) \in Q$, then by Corollary 3.4, there exist $f(e_1) \in T_2$ such that $f(X_1) \diamond f(e_1) = f(X_1) \Longrightarrow f(X_1 \divideontimes e_1) = f(X_1)$ which gives that $X_1 \divideontimes e_1 = X_1$. Finally, if $f(X_1) \in Q$ then by Corollary 3.3, we get $f(X_1) \diamond f(X_1) = f(e_1) \Longrightarrow X_1 \divideontimes X_1 = e_1$ where e_1 is the right identity of T_1. Therefore, $f^{-1}(Q)$ is a flower.

Corollary 3.5 The homomorphic image of a commutative flower is a commutative.

Proof: Let $f : (T_1, \divideontimes) \longrightarrow (T_2, \diamond)$ be a flowers homomorphism and let $(T_1, \divideontimes)$ be a commutative flower, then for any $X_1, y_1 \in T_1$ we have $X_1 \divideontimes y_1 = y_1 \divideontimes X_1$. By homomorphism we get $f(X_1 \divideontimes y_1) = f(y_1 \divideontimes X_1) \Longrightarrow f(X_1) \diamond f(y_1) = f(y_1) \diamond f(X_1)$ where $f(X_1), f(y_1) \in f(T_1)$.

Theorem 3.11 Let $(T_1, \divideontimes)$ and $(T_2, \diamond)$ be two flowers with their right identities. Then,

1. $T_1 \cong T_1 \times \{e_2\}$
2. $T_2 \cong T_2 \times \{e_1\}$

Proof: (1) Let $f^* : T_1 \longrightarrow T_1 \times \{e_2\}$ given by $f^*(X_1) = (X_1, e_2), X_1 \in T_1$. Then, we have to prove f^* is well-defined. If $X_1 = X_1'$ then $(X_1, e_2) = \left(X_1', e_2\right) = f^*(X_1) = f^*\left(X_1'\right)$ where $X_1' \in T_1$. This proved that f^* is well-defined. Now, $f^*\left(X_1 \divideontimes X_1'\right) = f^*\left(X_1 \divideontimes X_1', e_2\right) = f^*(X_1, e_2) \diamond f^*\left(X_1', e_2\right) = f^*(X_1) \diamond f^*\left(X_1'\right)$ which proved f^* is homomorphism. Next, if $f^*(X_1) = f^*\left(X_1'\right) \Longrightarrow (X_1, e_2) = \left(X_1', e_2\right) \Longleftrightarrow X_1 = X_1'$ which implies f^* is (1-1). Finally, for any $X_1 \in T_1$ we have $(X_1, e_2) \in T_1 \times \{e_2\}$ such that $f^*(X_1) = (X_1, e_2)$ which gives f^* is an onto. Hence, f^* is an isomorphism. Therefore, $T_1 \cong T_1 \times \{e_2\}$. Point two can be proved by the same way.

4. CONCLUSION

As a conclusion, this article proved the direct product of two Flowers is a Flower. Also, its generalization is form a flower. Moreover, this study proved that the direct product of two Flowers is commutative iff each one of them is commutative. Also its showed that the homomorphic image of a flower is a flower and the image (pre-image) of a sub-flower is also sub-flower. Also some related results concerned on the direct product and homomorphism of flower have been also discussed. For future work, we suggest the following works (i) some generalizations of the direct product and homomorphism of flower. (ii) the direct product of infinite flower

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CONFLICTS OF INTEREST

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