1. INTRODUCTION

Many studies have been considered the notation of the direct product and homomorphism of several algebraic structures. For instance, the direct product of an algebraic structure namely Bitonic algebras has been introduced by Ozbal [1]. The author generalized this notation to any finite collection of Bitonic algebras. He also proved that a finite family of Bitonic algebras is commutative if and only if each one of them is commutative. Furthermore, the author presented the homomorphism of the Bitonic algebras and investigated some related results about it. Setiani et. al [2] presented the concept of the direct product of BP-algebras and studied some of its properties. The authors extended this concept to a finite family of BP-algebras. Besides that, they proved the commutativity property of the direct product of BP-algebras. Moreover, they discussed the notation of the homomorphism in view of the concept of the direct product of BP-algebras. They showed some results about this notation and generalized these results to a finite family of BP-homomorphism. In addition, the notation of the direct product has been also studied for some other algebraic structures. For example, it has been provided for B-algebras by Angeline et. al [3] and for BG-algebras by Widianto et. al [4]. The direct product of BF-algebras has been introduced by Teves and Endam [5]. Angeline et. al [6] studied some mappings on the direct product of B-algebras. On the other hand, algebraic structures namely Flower, Garden and Farm have been introduced by Al-lahham [7]. He investigated some properties of these algebraic systems such as the commutativity property and the associativity property. Some of these algebraic systems have been extended by Atteya and Ressan [8]. They proved some results related to flower and garden such as the commutativity and some other properties. Also they determined some necessary and sufficient conditions for the structure \((\mathcal{T}, \ast)\) to be flower. Motivated by the works of the previous researchers, in this article we provided the concept of the direct product of Flower and the notation of the Flower homomorphism and studied some of their properties. This paper is structured as follows. In section two, some basic concepts that are needed in this research are stated. In section three, the main results of this paper are given. The conclusions of this paper and some suggestions for future works have been presented in section four.

2. BASIC CONCEPTS

This section contains some basic concepts that are needed in this research which presented as follows.

**Definition 2.1**[7] Let \(\mathcal{T} \neq \Phi\) then the binary operation \(\ast\) on \(\mathcal{T}\) is called ATL-law if for any \(x, y, z \in \mathcal{T}\) we have \(x \ast (y \ast z) = z \ast (y \ast x)\).

**Definition 2.2** [7] A pair \((\mathcal{T}, \ast)\) is called Flower if the axioms below are holds

1. \(x \ast (y \ast z) = z \ast (y \ast x), \forall x, y, z \in \mathcal{T}\).
2. \( \forall x \in T \exists e \in T \) (right identity of \( T \)) such that \( x * e = x \).
3. \( x * x = e \) (right inverses).

**Definition 2.3** [7] Let \( \Phi \neq \emptyset \subseteq T \) then \( \emptyset \) is sub-flower if it's a Flower with the binary operation of \( T \).

**Proposition 2.1** [7] Let \( (T, *) \) be a Flower. Then, \( (y * x) * x = (y * x) * x \) for any \( x, y, z \in T \).

**Proposition 2.2** [7] Let \( (T, *) \) be a Flower. Then, the points below are equivalent
1. \( T \) is Lahlhian group.
2. \( T \) has identity.
3. \( T \) is abelian.
4. \( T \) is an associative.

**Theorem 2.1** [8] Every flower is commutative.

### 3. MAIN RESULTS

This section deals with the notations of the direct product and homomorphism of flower. We start with the following definition.

**Definition 3.1** Let \( (T_1, *) \) and \( (T_2, \cdot) \) be two flowers. We define the direct product of \( T_1 \) and \( T_2 \) as a structure \( (T_1 \times T_2, \Box) \) such that all the points below are fulfilled.

1. \( (x_1, x_2) \Box (y_1, y_2) = (x_1 \cdot y_1, x_2 \cdot y_2) \) for any \( x_1, y_1 \in T_1 \) and \( x_2, y_2 \in T_2 \).
2. \( (x_1, x_2) \Box (y_1, y_2) = (x_2, y_2) \Box (y_1, y_2) \) for any \( x_1, y_1 \in T_1 \) and \( x_2, y_2 \in T_2 \).
3. \( (x_1, x_2) \Box (y_1, y_2) = (x_1, x_2) \Box (y_1, y_2) \) for any \( x_1, y_1 \in T_1 \) and \( x_2, y_2 \in T_2 \).

**Theorem 3.1** A structures \( (T_1, *) \) and \( (T_2, \cdot) \) are flowers iff \( (T_1 \times T_2, \Box) \) is a flower.

**Proof:** Suppose that \( (T_1, *) \) and \( (T_2, \cdot) \) are flowers to show that \( (T_1 \times T_2, \Box) \) is a flower. Since \( (T_1, *) \) and \( (T_2, \cdot) \) are flowers, then for each \( x_1, y_1 \in T_1 \) and \( x_2, y_2 \in T_2 \) we have the following points (1) \( (x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) = (x_1 * y_1, x_2 * y_2) \).

**Corollary 3.1** The direct product of finite family of flowers is a flower.

**Theorem 3.2** A system \( (T_1 \times T_2, \Box) \) is an abelian iff \( (T_1, *) \) and \( (T_2, \cdot) \) are abelian.

**Proof:** Let \( (T_1 \times T_2, \Box) \) is an abelian then for any \( (x_1, x_2), (y_1, y_2) \in T_1 \times T_2 \) we have that \( (x_1, x_2) \Box (y_1, y_2) = (y_1, y_2) \Box (x_1, x_2) \).

**Theorem 3.3** A finite intersection of flowers is a flower.

**Proof:** Let \( \{ (T_i, *) : i = 1, \ldots, v \} \) be a finite family of flowers to show that \( \bigcap_{i=1}^{v} (T_i, *) \) is a flower. Let \( x_i, y_i, z_i \in T_i \) for all \( i = 1, \ldots, v \). Since \( (T_i, *) \) is a flower for any \( i = 1, \ldots, v \), then we have \( x_i * (y_i * z_i) = x_i * (y_i * z_i) \) for any \( i = 1, \ldots, v \) which gives that condition one is hold. Since \( x_i \in \)
By Theorem 2.1, \( \bigcap_{i=1}^{v} (T_i \ast \#) \) then \( x_i \in (T_i \ast \#) \) for all \( i = 1, \ldots, v \). Since \( (T_i \ast \#) \) is a flower for any \( i = 1, \ldots, v \), there exist a right identities \( e_i \in (T_i \ast \#) \) for all \( i = 1, \ldots, v \) such that \( x_i \ast e_i = x_i \) for any \( i = 1, \ldots, v \). Thus, \( \bigcap_{i=1}^{v} (T_i \ast \#) \) has a right identity. Finally, we have \( x_i \in \bigcap_{i=1}^{v} (T_i \ast \#) \) which implies \( x_i \in (T_i \ast \#) \) for all \( i = 1, \ldots, v \). Since \( (T_i \ast \#) \) is a flower for any \( i = 1, \ldots, v \), then we have \( x_i \ast e_i = e_i \) for any \( i = 1, \ldots, v \). Thus, condition three is proved. Therefore, \((\bigcap T_i \ast \#): i = 1, \ldots, v) \) is a flower.

**Corollary 3.2** The finite intersection of a finite direct product of flowers is a flower.

**Proof:** Let \((T_i \ast \#): i = 1, \ldots, v\) be a finite family of flowers. By Corollary 3.1, \((\bigcap T_i \ast \#)\) is a flower. By Theorem 3.3, \((\bigcap T_i \ast \#)\) is a flower. Thus, \((\bigcap T_i \ast \#)\) is a flower. Therefore, as required.

**Theorem 3.4** Let \((T_1 \ast T_2)\) be a flower. Then, for each \((x_1, x_2), (y_1, y_2) \in T_1 \times T_2\), the following points are holds.

1. \((x_1, x_2) \ast ((x_1, x_2) \ast (y_1, y_2)) = (y_1, y_2)\)
2. \((x_1, x_2) \ast (x_1, x_2) = (e_1, e_2) \ast ((x_1, x_2) \ast (y_1, y_2))\)
3. \((x_1, x_2) \ast ((x_1, x_2) \ast (y_1, y_2)) = (e_1, e_2) \ast ((x_1, x_2) \ast (y_1, y_2))\)
4. \((x_1, x_2) \ast (y_1, y_2) = (x_1, x_2) \ast (y_1, y_2) = (e_1, e_2) \ast (y_1, y_2)\)
5. \((x_1, x_2) \ast (y_1, y_2) = (e_1, e_2) \ast (y_1, y_2) = (x_1, x_2) \ast (y_1, y_2)\)
6. \((x_1, x_2) \ast (y_1, y_2) = (x_1, x_2) \ast (y_1, y_2) = (e_1, e_2) \ast (y_1, y_2)\)
7. \((x_1, x_2) \ast (e_1, e_2) \ast (x_1, x_2) = (x_1, x_2) \ast (x_1, x_2) = (e_1, e_2)\)
8. \((x_1, x_2) \ast (x_1, x_2) \ast (e_1, e_2) = (x_1, x_2) \ast (x_1, x_2) = (e_1, e_2)\)

**Proof:** (1) \((x_1, x_2) \ast ((x_1, x_2) \ast (y_1, y_2)) = (x_1, x_2) \ast ((x_1, x_2) \ast (y_1, y_2)) = (x_1 \ast (x_1 \ast y_1), x_2 \ast (x_2 \ast y_2)) = (x_1 \ast y_1, x_2 \ast y_2)\) by ATL-law

\[ (y_1 \ast (x_1 \ast x_2), y_2 \ast (x_2 \ast x_3)) \]

(2) \((e_1, e_2) \ast ((x_1, x_2) \ast (y_1, y_2)) = (e_1, e_2) \ast ((x_1 \ast y_1, x_2 \ast y_2)) = (e_1 \ast (x_1 \ast y_1), e_2 \ast (x_2 \ast y_2)) = (y_1 \ast (x_1 \ast e_1), y_2 \ast (x_2 \ast e_2))\) by ATL-law

\[ (y_1 \ast x_1, y_2 \ast x_2) = (e_1, e_2) \ast (x_1, x_2) \ast (y_1, y_2) = (y_1, y_2) \]

(3) \((e_1, e_2) \ast (x_1, x_2) \ast (y_1, y_2) = (e_1, e_2) \ast (x_1, x_2) \ast (y_1, y_2) = (x_1 \ast x_1, x_2 \ast x_2) \ast (y_1, y_2)\)

\[ (x_1 \ast y_1, x_2 \ast y_2) \]

(4) \((x_1, x_2) \ast (y_1, y_2) = (x_1, x_2) \ast (y_1, y_2) = (x_1, x_2) \ast (y_1, y_2)\)

\[ (x_1 \ast y_1, x_2 \ast y_2) \]

(5) \((x_1, x_2) \ast (y_1, y_2) = (x_1, x_2) \ast (y_1, y_2) = (x_1, x_2) \ast (y_1, y_2)\)

\[ (x_1 \ast y_1, x_2 \ast y_2) \]

(6) \((x_1, x_2) \ast (y_1, y_2) \ast (x_1, x_2) = (x_1 \ast y_1, x_2 \ast y_2) \ast (x_1, x_2) = (x_1 \ast (y_1 \ast x_1), x_2 \ast (y_2 \ast x_2))\)

116

Let \( (T_s \times T_s, \otimes) \) be a flower. Then, for each \( (x_1, x_2), (y_1, y_2), (z_1, z_2) \in T_s \times T_s \), the following points are holds.

1. \( [(x_1, x_2) \otimes (y_1, y_2)] \otimes [(x_1, x_2) \otimes (z_1, z_2)] = (x_1, z_2) \otimes (y_1, y_2) \).

2. \( [(x_1, x_2) \otimes (y_1, y_2)] \otimes [(x_1, x_2) \otimes (z_1, z_2)] = (x_1, z_2) \otimes [(y_1, y_2) \otimes (z_1, z_2)] \).

3. \( [(x_1, x_2) \otimes (y_1, y_2)] \otimes [(z_1, z_2) \otimes (y_1, y_2)] = (x_1, x_2) \).

4. \( [(x_1, x_2) \otimes (y_1, y_2)] \otimes [(z_1, z_2) \otimes (y_1, y_2)] = (x_1, x_2) \).
Theorem 3.7 Let \((T_1 \times T_2, \Box)\) be a flower. If \((x_1, x_2) \Box (y_1, y_2) = (z_1, z_2) \Box (w_1, w_2)\) then \((y_1, y_2) = (x_1, x_2)\) for any \((x_1, x_2), (y_1, y_2), (z_1, z_2), (w_1, w_2) \in T_1 \times T_2\).

Proof: We have \((x_1, x_2) \Box (y_1, y_2) = (z_1, z_2) \Box (w_1, w_2)\) that is \((x_1 * y_1, x_2 * y_2) = (z_1 * w_1, z_2 * w_2)\). From this we get \(x_1 * y_1 = z_1 * w_1\) and \(z_2 * w_2 = z_2 * w_2\). We take the first equation which is \(x_1 * y_1 = z_1 * w_1\). Left multiply by \(x_1\) we obtain \(x_1 * (x_1 * y_1) = x_1 * (z_1 * w_1)\). Using AT-Law, we have \(y_1 \in (x_1 * x_2) = w_1 * (z_1 * x_2) \Rightarrow y_1 = (w_1 * z_1) * x_1\). By Proposition 2.1, we get \(y_1 = (w_1 * x_1) * z_1\). Replace \(w_1\) by \(x_1\) we have \(y_1 = x_1 * (x_1 * x_2)\). Using AT-Law, we have \(y_1 = z_1 * (x_1 * x_1) \Rightarrow y_1 = x_1\). By the same way we can prove that \(y_2 = z_2\). Therefore, \((y_1, y_2) = (x_1, x_2)\).

Theorem 3.8 If \((T_1 \times T_2, \Box)\) is an abelian group. Then, \((T_1 \times T_2, \Box)\) is a flower such that \((x_1, x_2) \Box (y_1, y_2) = (x_1, x_2) * (y_1, y_2)^{-1}\) for any \((x_1, x_2), (y_1, y_2) \in T_1 \times T_2\).

Proof: (1) For any \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in T_1 \times T_2\) then
\[
(x_1, x_2) \Box (y_1, y_2) \Box (z_1, z_2) = (x_1, x_2) \Box ((y_1, y_2) \Box (z_1, z_2))^{-1}
\]
\[
= (x_1, x_2) \Box ((y_1, y_2) \Box (z_1, z_2))^{-1}
\]
\[
= (x_1, x_2) \Box ((y_1, y_2) \Box (z_1, z_2))^{-1}
\]
(2) For any \((x_1, x_2) \in T_1 \times T_2\), there exist \((e_1, e_2) \in T_1 \times T_2\) such that \((x_1, x_2) \Box (e_1, e_2) = (x_1, x_2) \Box (e_1, e_2)^{-1} = (x_1, x_2) \Box (e_1, e_2) = (x_1, e_1, x_2 \Box e_2) = (x_1, x_2)\). (3) For any \((x_1, x_2) \in T_1 \times T_2\) we have \((x_1, x_2) \Box (e_1, e_2) = (x_1, x_2) \Box (e_1, e_2)^{-1} = (x_1, x_2) \Box (e_1, x_2) = (x_1, x_2)\).

Definition 3.2 Let \((T_1, \star)\) and \((T_2, \star')\) be two flowers. We define the flowers homomorphism as a mapping \(f: T_1 \rightarrow T_2\) in which \(f(x * y) = f(x) \star f(y)\) for any \(x, y \in T_1\).

Remark 3.1 Let \(f: T_1 \rightarrow T_2\) be a flowers homomorphism. If \(f\) is an onto, then it's called an epimorphism, if \(f\) is (1-1) then it's called monomorphism and if \(f\) is bijection then it's called isomorphism.

Theorem 3.9 The homomorphic image of a flower is a flower.

Proof: Let \(f: (T_1, \Box) \rightarrow (T_2, \Box)\) be a flowers homomorphism to show that \((T_1, \Box)\) is a flower. Let \(x_1, y_1, z_1 \in T_1\) then \(f(x_1), f(y_1), f(z_1) \in f(T_1)\). Since \(x_1, y_1, z_1 \in T_1\) and \((T_1, \Box)\) is a flower, then we get \(x_1 * (y_1 * z_1) = x_1 * (y_1 * z_1)\). Since \(f\) be a homomorphism, then \(f(x_1) * f(y_1) = f(x_1) * f(y_1)\) which implies \(f(x_1) * f(y_1) = f(x_1) * f(y_1)\). Thus, condition one is hold. Since \(x_1 \in T_1\) and \((T_1, \Box)\) is a flower, then there exist a right identity \(e_1 \in T_1\) such that \(x_1 * e_1 = x_1\). By homomorphism of \(f\), we have \(f(x_1) * f(e_1) = f(x_1) * f(e_1)\) which proves condition two. Finally, since \(x_1 \in T_1\) and \((T_1, \Box)\) is a flower, then we have \(x_1 * x_1 = e_1\). This gives that \(f(x_1 * x_1) = f(e_1) \Rightarrow f(x_1) * f(x_1) = f(e_1)\). So, condition three is hold. Therefore, \((T_1, \Box)\) is a flower.

Corollary 3.4 Let \(f: (T_1, \star) \rightarrow (T_2, \star)\) be a flowers homomorphism. If \(e_1\) is the right identity of \((T_1, \Box)\), then \((T_2, \Box)\) is the right identity of \((T_2, \Box)\).

Proof: Let \(e_1\) be a right identity of \((T_1, \Box)\), then for any \(x_1 \in T_1\) we have \(x_1 * e_1 = x_1 \Rightarrow f(x_1) * f(e_1) = f(x_1)\) which gives that \((T_2, \Box)\) is the right identity of \((T_2, \Box)\).

Theorem 3.10 Let \(f: (T_1, \Box) \rightarrow (T_2, \Box)\) be a flowers homomorphism. Then,
1. If \(\Omega\) is a sub-flower of \(T_1\), then \(f(\Omega)\) is a sub-flower of \(T_2\).
2. If \(\Omega\) is a sub-flower of \(T_2\), then \(f^{-1}(\Omega)\) is a sub-flower of \(T_1\).
Proof: (1) Let \( \mathcal{Q} \) be a sub-flower of \( \mathcal{T}_1 \), then by Definition 2.3, we have \( \mathcal{Q} \) is a flower. By Theorem 3.9, \( f(\mathcal{Q}) \) is a flower of \( \mathcal{T}_2 \). (2) Let \( f^{-1}(\mathcal{Q}) = \{x_i \in \mathcal{T}_2 : f(x_i) \in \mathcal{Q} \} \). That is mean \( f(x_i) \in \mathcal{Q} \Rightarrow x_i \in f^{-1}(\mathcal{Q}) \). Since \( \mathcal{Q} \) is a sub-flower of \( \mathcal{T}_2 \), then by Definition 2.3, \( \mathcal{Q} \) is a flower. This gives us that if \( f(x_i), f(y_i) \in \mathcal{Q} \) where \( x_i, y_i, x_1 \in f^{-1}(\mathcal{Q}) \) we have that \( f(x_i) \circ f(y_i) = f(x_i \circ y_i) = f(x_i \circ y_i) = f(x_i) \circ f(y_i) \). This condition one is hold. For condition two, if \( f(x_i) \in \mathcal{Q} \), then by Corollary 3.4, there exist \( f(e_1) \in \mathcal{T}_2 \) such that \( f(x_i) \circ f(e_1) \neq f(x_i) \). Therefore, \( f(\mathcal{Q}) \) is a flower. ■

Corollary 3.5 The homomorphic image of a commutative flower is a commutative.

Proof: Let \( (\mathcal{T}_1, \#) \to (\mathcal{T}_2, \#) \) be a flowers homomorphism and let \( (\mathcal{T}_2, \#) \) be a commutative flower, then for any \( x_1, y_1 \in \mathcal{T}_1 \) we have \( x_1 \# y_1 = y_1 \# x_1 \). By homomorphism we get \( f(x_1 \# y_1) = f(y_1 \# x_1) \Rightarrow f(x_1) \# f(y_1) = f(y_1) \# f(x_1) \) where \( f(x_i), f(y_i) \in f(\mathcal{T}_1) \). ■

Theorem 3.11 Let \( (\mathcal{T}_1, \#) \) and \( (\mathcal{T}_2, \#) \) be two flowers with their right identities. Then,
1. \( \mathcal{T}_1 \cong \mathcal{T}_1 \times \{e_2\} \)
2. \( \mathcal{T}_2 \cong \mathcal{T}_2 \times \{e_1\} \).

Proof: (1) Let \( \mathcal{T}_1 \to \mathcal{T}_1 \times \{e_2\} \) given by \( f(x_i) = (x_i, e_2), x_1 \in \mathcal{T}_1 \). Then, we have to prove \( f^* \) is well-defined. If \( x_1 = x_1 \) then \( (x_1, e_2) = (x_1, e_2) = f^*((x_i)) \) where \( x_1 \in \mathcal{T}_1 \). This proved that \( f^* \) is well-defined. Now, \( f^*(x_1 \# x_2) = f^*(x_1, e_2) \circ f^*(x_2, e_2) = f^*(x_1) \circ f^*(x_2) \) which proved \( f^* \) is homomorphism. Next, if \( f^*(x_i) = f^*(x_i) \) \( \Rightarrow (x_i, e_2) = (x_i, e_2) \) which implies \( f^* \) is (1-1). Finally, for any \( x_1 \in \mathcal{T}_1 \) we have \( (x_1, e_2) \in \mathcal{T}_1 \times \{e_2\} \) such that \( f^*(x_i) = (x_i, e_2) \) which gives \( f^* \) is an onto. Hence, \( f^* \) is an isomorphism. Therefore, \( \mathcal{T}_1 \cong \mathcal{T}_1 \times \{e_2\} \). Point two can be proved by the same way. ■

4. CONCLUSION

As a conclusion, this article proved the direct product of two Flowers is a Flower. Also, its generalization is a form a flower. Moreover, this study proved that the direct product of two Flowers is commutative if each one of them is commutative. Also its showed that the homomorphic image of a flower is a flower and the image (pre-image) of a sub-flower is also sub-flower. Also some related results concerned on the direct product and homomorphism of flower have been also discussed. For future work, we suggest the following works (i) some generalizations of the direct product and homomorphism of flower. (ii) the direct product of infinite flower

5. REFERENCES