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On θg^{**}- Closed Sets in Topological Spaces

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ABSTRACT: In this paper we have introduced a new class of closed in topological spaces called θg^{**}-closed sets and study some of its properties . Further we introduce the concept θg^{**}-continuous functions , θg^{**}-irresolute functions . As an application we introduce two news paces namely . $T\theta^{**}$-space, $^{**}T\theta$- space .Further, θg^{**}-continuous, and θg^{**}-irresolute mappings are also introduced and investigated .

Keywords: θ g ** closed sets, θ g ** continuous functions, θ g ** irresolute functions, T θ ** space, ** T θ space

1. INTRODUCTION

In 1970, Levine [1] presented the g-closed set class. In 1994, Maki.et.al [2] defined αg - closed sets. In 1990, Arya and Tour [3] introduced the concept of gs-closed sets. The terms gsp-closed sets, gpr-closed sets, and rg-closed sets were first introduced by Dontchey [4], Gnanambal [5], Palaniappan, and Rao [6]. Veerakumar [7] introduced g^*-closed sets in 1991.J.Dontchey [4] introduced gsp-closed set in 1995. Levine [8] $T_{1/2}$-space, Tb-space, and T_b - space were all introduced by Devi et al. [9,10] [9]. Veerakumar [7]introduced $T_{1/2}{}^{**}$ - space. This essay's goal is to introduce and study the notions of θg^{**} -closed sets, θg^{**}-continuous map, θg^{**}-irresolute maps, $T\theta^{**}$-space**,** $^{**}T\theta$- space

2. BASIC CONCEPTS

Definition 1.1

Let (X, τ) be a topological space. A subset B of the space X is called

1) A generalized closed set (brifly , g - closed) [1]if cl(B) $\subseteq$ V , every time B $\subseteq$ V and V are open .
2) A semi – generalized closed set (brifly ,sg – closed) [10] if scl(B) $\subseteq$ V, every time B $\subseteq$ V and V was semi – open .
3) A generalized semi- closed set (brifly, gs – closed) [11] if Scl(B) $\subseteq$ V, every time B $\subseteq$ V and V are open .
4) A generic semi - preclosed set (brifly , gsp – closed) [12] if Spcl (B) $\subseteq$ V, every time B $\subseteq$ V and V are open .
7) regular generic closed set (brifly , r– g – closed) [6] if cl(B) $\subseteq$ V, every time B $\subseteq$ V and V are regular open

Definition 1.2

A topological space (X , τ) is said to be

1) a $T_{1/2}$ space [1] if every g – closed set in it is closed .
2) a T_b space [9] if every gs – closed set in it is closed .
3) a T_d space [9] if every gs – closed set in it is g – closed .
4) an αT_d space [13] if every αg – closed set in it is g – closed .
5) an αT_b space [13] if every αg – closed set in it is closed .
6) a $T_{1/2}{}^*$ space [7] if every g^* - closed set in it is closed .
7) $^*T_{1/2}$ space [7] if every g – closed set in it is g^* - closed .

3. θG**-CLOSED SETS

In this section we introduce and study the notion of θg^{**}-closed sets in topological spaces and obtain some of its basic properties .

Definition 2.1

A subset A of atopological space (X , τ) is called θg^{*}-closed sets if Clθ(A) ⊆ U and U is g - open in (X , τ) .

Definition 2.2

A subset A of atopological space (X , τ) is called θg^{**}-closed sets if Clθ(A) ⊆ U and U is g^{*} - open in (X , τ) .

Theorem 2.3

Every r –closed set is θg^{**}-closed sets .

Proof :

Suppose that A be a r –closed set in X .

Let U be a g^{*}-open set such that A ⊆ U .

Since A is r –closed, then we have rcl(A) = A ⊆ U .

But Clθ(A) ⊆ $rcl\,(A)$⊆ U .

Therefore Clθ(A) ⊆ U .

Hence A is a θg^{**}-closed set .

But the convers of Theorem (2.3) is not true .

Example 2.4

Let X = { 1 , 2 , 3 }, τ = { X , ∅,{ 1},{2},{1,2},{2,3}},

r –closed = {X, ∅, (2, 3}, {1, 3}} *and*

θg^{**}-closedset ={X, ∅, (1}, (1, 3}, {2, 3}} .

Let A = {1}.

Then the subset A is θg^{**}-closed but not a r –closed set .

Theorem 2.5

The union of two θg^{**}-closed subsets are θg^{**}-closed .

Proof :

Let A and B any two θg^{**}-closed sets in X .

Such that A ⊆ U *and* $B \subseteq U$ where U is g^{*}-open in X and so

A ∪B⊆ U .

Since A and B are θg^{**}-closed .

A ⊆ Clθ(A) and B ⊆ Clθ(B) and hence A ∪B ⊆ $Cl\theta\,(A)$ ∪ Clθ(B) ⊆ Clθ(A ∪B) .

Thus A ∪B *is* θg^{**}-closed set in (X , τ) .

Example 2.6

Let X = { 1 , 2 , 3 }, τ = { X ,∅,{2,{3},{2,3}} and

θg^{**}-closed = {X, ,∅, (1}, (3}, (1, 2}, {1, 3}}.

Let A = {1} and B = {3}, then A ∪B = (1, 3} is also θg^{**}-closed .

Theorem 2.7

The intersection of two θg^{**}-closed subset are θg^{**}-closed.

Proof :

Let A and B any two θg^{**}-closed sets in X .

Such that A ⊆ U *and* $B \subseteq U$ where U is g^{*}-open in X and so

A ∩B⊆ U .

Since A and B are θg^{**}-closed .

A ⊆ Clθ(A) and B ⊆ Clθ(B) and hence A ∩B ⊆ $Cl\theta\,(A)$ ∩ Clθ(B) ⊆ Clθ(A ∩ B) .

Thus A ∩B *is* θg^{**}-closed set in (X , τ) .

Example 2.8

Let X = { 1 , 2 , 3 }, τ = { X ,∅,{2},{3},{2,3},{1,2}} and

θg^{**}-closed = {X, ,∅, (1}, (3}, (1, 2}, {1, 3}}.

Let A = {1,2} and B = {1,3}, then A ∩B = (1} is also θg^{**}-closed .

Proposition 2.9

Every θg^{**} -closed set is gpr-closed .

Proof :

Let A ⊆ U where U is regular open .

Then U is θg^{*}-open .

Since A is θg^{**} - closed , cl(A) ⊆ U ,

which implies pcl(A) ⊆ Cl(A) ⊆ U .
Therefore A is gpr – closed .
But the convers of (Propostion 2.9) is not true .

Example 2.10

Let X = { 1 , 2 , 3 }, τ = { X , ∅,{2},{1,2}},
gpr –closed = {X, ∅, (1} , (2} , (3} , (1, 2} , (2, 3} , {1, 3}} *and*
θg^{**}-closedset ={X, ∅, (3} , (1, 3} , {2, 3}} .
Let A = {1}. Then the subset A is gpr -closed but not a θg^{**} –closed set .

Proposition 2.11

Every θg^{**}- closed set is gs – closed .

Proof :

Let A be a θg^{**}-closed set .
Let A ⊆ U , and U be open .
Then $cl(A) \subseteq U$, since U is θg^{*}-open and A is θg^{**}-closed .
But $scl(A) \subseteq cl(A) \subseteq U.$
Hence A is gs-closed .
But the convers of (Propostion 2.11) is not true .

Example 2.12

Let X = { 1 , 2 , 3 }, τ = { X , ∅,{2},{1,2}},
gs –closed = {X, ∅, (2} , (3} , (1, 2} , (2, 3} , {1, 3}} *and*
θg^{**}-closedset ={X, ∅, {2, 3}} .
Let A = {1,2}.
Then the subset A is gs -closed but not a θg^{**} –closed set .

Proposition 2.13

Every θg^{**}- closed set is rg – closed .

Proof :

Let A be a θg^{**}-closed set .
Let A ⊆ U and U be regular open .
Then U is θg^{*} -open and hence U is θg^{*}-open .
since A is θg^{**}- closed , Cl(A) ⊆ U .
Therefore A is rg – closed .
But the convers of (Proposition 2.13) is not true .

Example 2.14

Let X = { 1 , 2 , 3 }, τ = { X , ∅,{2},{1,2}},
rg –closed = {X, ∅, (1} , (2} , (3} , (1, 2} , (2, 3} , {1, 3}} *and*
θg^{**}-closedset ={X, ∅, (3} , (1, 3} , {2, 3}} .
Let A = {1}.
Then the subset A is rg -closed but not a θg^{**} –closed set .

Proposition 2.15

Every θg^{**}- closed set is g – closed .

Proof :

Let A be a θg^{**}-closed set
Let A ⊆ U and U be open then U is θg^{*} open .
Since A is θg^{**} - closed , Cl(A) ⊆ U .
Therefore A is g – closed .
But the convers of (Propostion 2.15) is not true .

Example 2.16

Let X = { 1 , 2 , 3 }, τ = { X , ∅,{1},{1,2}},
g–closed = {X, ∅, (2} , (3} , (1, 2} , (2, 3} , {1, 3}} *and*
θg^{**}-closedset ={X, ∅, {2, 3}} .
Let A = {1,2}.
Then the subset A is gs -closed but not a θg^{**} –closed set .

Proposition 2.17

Every θg^{**}- closed set is gp – closed .
But the convers of (Proposition 2.17) is not true .

Example 2.22

Let X = { 1 , 2 , 3 }, τ = { X , ∅,{2},{1,2}},
gp –closed = {X, ∅, (2} , (3} , (1, 2} , (2, 3} , {1, 3}} *and*
θg^{**}-closedset ={X, ∅, {2, 3}} .
Let A = {1,2}.
Then the subset A is gp - closed but not a θg^{**} –closed set .

4. θG^{**}-CONTINUOUS AND θG^{**}-IRRESOLUTE MAPS

Difinition 3.1

A function (X , τ) → (Y , μ) is called θg^{**} - countinuous if f^{-1} (V) is a θg^{**} - closed set of (X , τ) for every closed set V of (Y , μ) .

Definition 3.2

A function f : (X , τ) → (Y , μ) is called θg – irresolute if f^{-1} (V) is a θg – closed set of (X , τ) for every θg – closed set V of (Y , μ) .

Definition 3.3

A function f : (X , τ) → (Y , μ) is called θg^{**} – irresolute if f^{-1} (V) is a θg^{**} – closed set of (X , τ) for every θg^{**}– closed set V of (Y , μ) .

Theorem 3.4

Every continuous map is θg^{**}- continuous .

Proof :

Let f :(X , τ) → (Y , μ) be continuous and let F be any closed set of Y .
Then f^{-1}(F) is closed in X .
Since every closed set is θg^{**}- closed , f^{-1} (F) is θg^{**} closed .
Therefore f is θg^{**} -continuous .
But the convers of (Theorem 3.4) is not true .

Example 3.5

Let X = Y = { 1 , 2 , 3 } , τ = { X , ∅ , { 1 },{1,2} } , μ = { Y , ∅ , {3 } } ,
f (X , τ) → (Y , μ) is defined by f(1) = 2, f(2) = 1, f(3) = 3 .
Then f is θg^{**} -continuous but not continuous .
Since for the closed set {1,2} in Y .
f^{-1}({1,2}) = {1,2} is θg^{**} -continuous but not continuous .

Theorem 3.6

Every θg^{**}- continuous function is rg –continuous .

Proof :

Let f : (X , τ) → (Y , μ) be a θg^{**}- continuous function .
Let V be closed set of (Y , μ) .
Since f is θg^{**}- continuous , then f^{-1}(V) is a θg^{**} - closed set in (X , τ).
Since every θg^{**} - closed set is rg –closed .
f^{-1}(V) is rg –closed set in (X , τ) .
Therefore f is rg –continuous .
But the convers of (Theorem 3.6) is not true .

Example 3.7

Let X = Y = { 1 , 2 , 3 } , τ = { X , ∅ , { 1 },{2},{1,2}} , μ = { Y , ∅ , {3 }} .
Let the function f : (X , τ) → (Y , μ) is defined by
f(1) = 2, f(2) = 1, f(3) = 3 .
Then f is rg –continuous but not θg^{**} -continuous .
Since for the closed set {1,2} in Y .
f^{-1}({1,2}) = {1,2} is rg –closed but not θg^{**} -closed set in (X , τ) .

Theorem 3.8

Every θg^{**}- continuous function is gpr –continuous .

Proof :

Let f : (X , τ) → (Y , μ) be a θg^{**} -continuous function .
Let V be closed set of (Y , μ) .
Since f is θg^{**} - continuous , then f^{-1} (V) is a θg^{**} - closed set in (X , τ) .
By (proposition 4.1.) f^{-1} (V) is gpr –closed set of (X , τ) .
the convers of the above (Theorem 3.8) is not true .

Example 3.9

Let X = Y = { 1 , 2 , 3 } , τ = { X , ∅ , { 1 },{3},{1,3}} ,
μ = { Y , ∅ , {2 }} .
Let the function f : (X , τ) → (Y , μ) is defined by
f(1) = 3, f(2) = 2, f(3) = 1 .
Then f is gpr –continuous but not θg^{**} -continuous .
Since for the closed set {1,3} in Y .
f^{-1}({1,3}) = {1,3} is gpr –closed but not θg^{**} -closed set in (X , τ).

Theorem 3.10

Every θg^{**}- continuous function is gs –continuous .

Proof

Let f : (X , τ) → (Y , μ) be a θg^{**}- continuous function .
Let V be closed set of (Y , μ) .
Since f is θg^{**}- continuous , then f^{-1}(V) is a θg^{**} - closed set in (X , τ) .
Since every θg^{**} - closed set is gs –closed .
f^{-1}(V) is gs –closed set in (X , τ) .
Therefore f is gs –continuous .
But the convers of (Theorem 3.10) is not true .

Example 3.11

Let X = Y = { 1 , 2 , 3 } , τ = { X , ∅ , { 1 },{1,3}} , μ = { Y , ∅ ,{2 },{1,2}} .
Let the function f : (X , τ) → (Y , μ) be an identity function .
Then f is gs –continuous but not θg^{**} -continuous .
Since for the closed set {1,3} and {3} in Y .
f^{-1}({1,3}) = {1,3}and f^{-1}{3} = {3}is gs –closed but not θg^{**} -closed set in (X , τ) .

Theorem 3.12

Every θg^{**}- continuous function is gp –continuous .

Proof :

Let f : (X , τ) → (Y , μ) be a θg^{**} -continuous function .
Let V be closed set of (Y , μ) .
Since f is θg^{**} - continuous , then f^{-1} (V) is a θg^{**} - closed set in (X , τ) .
By (proposition 4.1.) f^{-1} (V) is gp –closed set of (X , τ) .
the convers of the above (Theorem 3.12) is not true .

Example 3.13

Let X = Y = { 1 , 2 , 3 } , τ = { X , ∅ , { 1 },{1,2}},
μ = { Y , ∅ ,{1,3}} .
Let the function f : (X , τ) → (Y , μ) be an identity function .
Then f is gp –continuous but not θg^{**} -continuous .
Since for the closed set {2} in Y .
f^{-1}({2}) = {2} is gp –closed but not θg^{**} -closed set in (X , τ) .

Theorem 3.14

Every θg^{**} -irresolute is θg^{**} -continuous .

Proof :

Let f :(X , τ) → (Y , μ) be a θg^{**} -irresolute .
Let V be a closed set of (Y , μ) .
Then V is θg^{**} -closed and f^{-1}(V) is θg^{**} -closed .
Since f is a θg^{**} -irresolute .
Hence f is θg^{**} -continuous .
the convers of the above (Theorem 3.14) is not true .

Example 3.15

Let X = Y = { 1 , 2 , 3 } , τ = { X , ∅ , {3}} ,
μ = { Y , ∅ ,{1},{2,3}} .
Let the function f : (X , τ) → (Y , μ) is defined by
f(1) = 3, f(2) = 2, f(3) = 1 .
Then f is gpr –continuous but not θg^{**} -continuous .
Since for the closed set {1,3} in Y .
f^{-1}({1,3}) = {1,3} is gpr –closed but not θg^{**} -closed set in (X , τ).

Proposition 3.16

Every θg^{*} - continuous map is θg^{**} - continuous .

Proof :

Let f : (X , τ) → (Y , μ) be θg^{**} - continuous .

let V be closed set of Y .

Then $f^{-1}(V)$ is θg^{*} - closed and hence by propositio (3.1.2)

it is θg^{**} - closed .

Hence f is θg^{**}- continuous .

But the convers of (Propostion 3.16) is not true .

Let X = Y = { 1 , 2 , 3 } , τ = { ∅ , X ,{ 1 } },

μ = { ∅ , X , { 2 } } .

Let f : (X , τ) → (Y , μ) be the identity map . A = { 1 , 3 } is closed in (Y , μ) and is θg^{**} - closed in (X , τ) but not θg^{*} - closed in (X , τ) .

Therefore f is θg^{**}- continuous but not θg^{*} - continuous .

5. APPLICATIONS OF θG**-CLOSED SETS

Definition 4.1

A space (X , τ) is called a $T\theta^{*}$- space if every θg^{*}- closed set is closed .

Definition 4.2

A space (X , τ) is called a $T\theta^{**}$- space if every θg^{**} - closed set is closed.

Theorem 4.3

Every $T_{1/2}$ - space is $T\theta^{**}$- space .

Proof :

Let (X , τ) be a $T_{1/2}$ – space .

Since every θg^{**} -closed set is g-closed, A is g-closed .

Since (X , τ) is a $T_{1/2}$ – space, A is closed .

Hence (X , τ) is a $T\theta^{**}$- space .

But the convers of (theorem 4.3.4) is not true .

Example 4.4

Let X ={1, 2, 3, 4}, $\tau = (X,\ \emptyset,\ (1\},\{1,\ 2\}\}$.

(X, τ) *is a* $T\theta^{**}$- space but not a $T_{1/2}$ - space since A ={1, 3} is g-closed but not closed

and hence it is not a $T_{1/2}$ - space .

Hence a $T\theta^{**}$- space need not be a $T_{1/2}$ – space

Theorem 4.5

Every T_b- space is a $T\theta^{**}$- space .

the convers need of the above (Theorem 4.5) is not true .

Example 4.6

Let X = {1, 2, 3}, $\tau = (X,\ \emptyset,\ (1\},(2\},(1,\ 2\}\}$.

(X , τ) *is a* $T\theta^{**}$- space but not a T_b- space since A = {1} is

gs-closed but not closed and hence it is not a T_b - space .

Hence a $T\theta^{**}$- space need not be a T_b – space .

Definition 4.7

A space (X , τ *is called an* $**T\breve{\theta}$ *space if every* θg^{**}-closed set of

(X , τ) *is a* θg^{*}-closed set .

Example 4.8

Let X ={1, 2, 3}, $\tau = (X,\ \emptyset,\ (1\},\ (1,\ 2\}\}$.

(X , τ) *is* $^{**}T\theta$ – space but not a $T\theta^{*}$- space since A = {1, 3} is

θg^{*}-closed but not closed .

Example 4.9

Let X = {1, 2, 3}, $\tau = (X,\ \emptyset,\ (1\}\}$.

(X , τ) *is* $T\theta^{*}$- space but not a $^{**}T\theta$ – space since A = {1,3} is

θg^{**}-closed but not θg^{*}-closed .

Theorem 4.10

Every $T\theta^{**}$- space is $^{**}T\theta$ – space .

Proof :

Let (X , τ) *is* Tθ^{**}- space .
Let A be a θg**-closed set of (X , τ) .
Since (X , τ) *is a* Tθ^{**}- space , A is closed .
By theorem (4.1.), A is θg*-closed .
Therefore (X , τ) *is a* **Tθ – space .
the convers of the above (Theorem 4.10) is not true .

Example 4.11

In example (4.3.12), (X , τ) *is a* **Tθ – space but not a Tθ^{**} - space since A = {1, 3} is θg**-closed but not closed .

Theorem 4.12

Every T$_b$ – space is a **Tθ – space .

Proof :

Let (X , τ) *is* T$_b$ – space .
Then by theorem(4.3.6), it is a Tθ^{**} - space .
Therefore by theorem(4.3.13), it is a **Tθ – space .
the convers of the above (Theorem 4.12) is not true .

Example 4.13

In example(4.3.11), (X , τ) *is* **Tθ – space but not a T$_b$ –space since
A = {1, 3} is gs-closed but not closed. [14, 15]

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The author declares no conflict of interest.

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