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On BDM-Algebras

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ABSTRACT: Abstract algebra is one of the influential branches in the field of pure Mathematics. This field concentrate on the study of the algebraic structures and discussed the relationships among them. Many studies have been presented various types of algebraic structures some of which independently and some others have been constructed via extending form other algebraic structure in order to investigate some of their properties. In this paper, we established an algebraic structure namely BDM-Algebras and studied some of its properties. Furthermore, we presented the 0-commtativity, sub-algebra and normal sub-algebra of a BDM-Algebras. In addition, we provided BDM-homomorphism and the kernel of BDM-homomorphism with some properties of them. Moreover, we introduced the quotient BDM-Algebras by using the notation of normal ideal of BDM-Algebras. Finally, we introduced the concept of the direct product of BDM-Algebras and some of its properties have been discussed. Some examples are given to illustrated the results.

Keywords: Balgebras, BDalgebras, BMalgebras, BFalgebras, BG algebras and BEalgebras

1. INTRODUCTION

An algebraic structure namely B-algebras has been introduced by Neggers and Kim [1]. This algebraic structure has been defined as a triple $(\mathcal{X}, \ominus, 0)$ such that it obeys some specific conditions. The authors presented the commutativity of this algebraic structure and proved some results about it. Another algebraic structure namely BMAlgebras which based on some conditions of a B-Algebras has been introduced by C. B. Kim and H. S. Kim [2]. They showed that some important results that concern on this algebraic structure. In same year, H. S. Kim and Y. H. Kim [3] presented an algebraic structure namely BE-algebras and studied some of its properties. Furthermore, the notation of BF-Algebras has been introduced by A. Walendziak [4]. The author established this algebraic structure by considering some conditions from a B-Algebras which are (B1) and (B2). The author presented the notations of ideal and normal ideal in BFAlgebras. Besides, the author also provided the BF-homomorphism and studied some of its properties and by using the notation of normal ideal, the author presented the quotient BF-Algebras. One year later, new algebras structure namely BG-Algebras has been introduced by C. B. Kim and H. S. Kim [5]. This algebraic structure has been extended from a B-Algebras by considering two conditions from it which are (B1) and (B2). The authors discussed many important results for a BG-Algebras such as subalgebra of BG-Algebra and normal sub-algebra of BG-Algebra and they proved that any normal sub-algebra of a BG-Algebra is a BG-sub-algebra of a BG-Algebra but not conversely. Besides that, they investigated the BG-homomorphism and discussed many results of it such as the kernel of a BG-homomorphism. In addition, they introduced the quotient BG-Algebras by using the notation of normal sub-algebras of BG-Algebras. Very recently, an algebraic structure namely BD-Algebras has been introduced by Mahdi and Nouri [6]. The authors presented some properties of this algebraic structure such as an ideal, p-ideal, q-ideal, Bd1-ideal and Bd2-ideal of a BD-Algebras. Moreover, they showed that any Bd1-ideal in a BD-Algebras is an ideal. Also, any Bd2-ideal in associative BD-Algebras is an ideal. Furthermore, they

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studied some other properties among p-ideal and q-ideal. Motivated by the work of previous researchers, a class of an abstract algebra namely BDM-Algebras with some of its properties have been introduced in this paper. This article has been structured as follows: Some related results from the work of the previous researchers have been provided in section two. Section three contains the notation of a BDM-Algebras with some of its properties. In section four, the conclusions of the present paper are given.

2. BASIC CONCEPTS

This section includes some previous results that are needed in this study which are given as follows.

Definition 2.1 [1] A system $(\Psi, \ominus, 0)$ is called B-algebras, if it satisfying the following axioms

1. $\mathcal{F} \ominus r = 0$,
2. $\mathcal{F} \ominus 0 = x$,
3. $(\mathcal{F} \ominus t) \ominus s = r \ominus (s \ominus (0 \ominus t))$ for any $r, t, s \in \mathfrak{R}$.

Definition 2.2 [2] A system $(\mathfrak{R}, \ominus, 0)$ is called BM-algebras, if it satisfying the following axioms

1. $\mathcal{F} \ominus 0 = \mathcal{F}$,
2. $(\mathcal{F} \ominus t) \ominus (f \ominus s) = (s \ominus t)$ for any $r, t, s \in \mathfrak{R}$.

Definition 2.3 [3] Let $(\Psi, \ominus, 0)$ be a BE-algebra and let $\Phi \neq\subseteq \Psi$, then Ψ is called filter of $\mathcal{R}$ if
1.1 $\in \Psi$,

2. $r \ominus t \in$ and $r \in\Longrightarrow t \in$.

Definition 2.4 [6] A system $(\mathfrak{R}, \ominus, 0)$ is called BD-algebras, if it satisfying the following axioms

1. $\mathbf{r} \ominus 0 = \mathbf{r}$
2. $r \ominus r = r$,
3. $(r \ominus t) \ominus r = (0 \ominus r) \ominus t$ for any $r, t, s \in$ K.

3. BDM-ALGEBRAS

This section contains the main results of this paper. We begin with the following definition.

Definition 3.1 A triple $(\Psi, \ominus, 0)$ is called BDM-Algebras, if for any $r, t, s \in \Psi$ the conditions below are holds.

1. $r \ominus r = r$,
2. $f \ominus 0 = r$,
3. $(\mathrm{r} \ominus \mathrm{t}) \ominus (\mathrm{r} \ominus \mathrm{s}) = (\mathrm{s} \ominus \mathrm{t})$.

Example 3.1 Let $\mathbb{X} = \{0, 1, 2, 3\}$ and $\ominus$ is a binary operation defined in the following Cayley table

$\ominus$	0	1	2	3
0	0	2	1	3
1	1	1	3	2
2	2	3	2	1
3	3	1	2	3

Then, $(\Psi, \ominus, 0)$ is a BDM-Algebras.

Example 3.2 Let $\mathbb{K} = \{0, 1, 2\}$ and $\ominus$ is a binary operation defined in the following Cayley table

$\ominus$	0	1	2
0	0	2	1
1	1	1	2
2	2	1	2

Then, $(\Psi, \ominus, 0)$ is a BDM-Algebras.

Remark 3.1 Throughout this paper we meant by Ψ is a BDM-algebras unless otherwise we mentioned.

Definition 3.2 Let $\Phi \neq Ж \subseteq Ѫ$, then Ж is called BDM-sub-algebra, if for any $r, s \in$ we have $r \ominus s \in$.

Example 3.3 Consider a BDM-Algebras which is given in Example 3.1. Let $\Psi_1 = \{0, 3\}$ and $\Psi_2 = \{0, 1, 2\}$ then $\mathscr{W}_1$ is a BDM-sub-algebra of $\mathcal{X}$ but Ψ_2 is not, because $1 \ominus 2 = 3 \notin \Psi_2$

Definition 3.3 Let $\Phi \neq Ж \subseteq Ѫ$, then Ж is called filter of Ѫ, if it obeys the following axioms

1.0 $\notin$,

2. If $x \ominus s \in$ and $r \in \mathcal{W} \Rightarrow s \in \mathcal{H}$.

Example 3.4 Consider a BDM-Algebras which is given in Example 3.1. Let $= \{1, 2, 3\}$, then $\mathcal{W}$ is a filter of $\boldsymbol{X}$.

Remark 3.2 A filter of a BDM-Algebras is a BDM-sub-Algebra but the convers is not necessary true.

Example 3.5 A filter in Example 3.4 is a BDM-sub-Algebra of $\mathbb{Z}$. However, the BDMsub-Algebra given in Example 3.3 is not filter of $\mathfrak{R}$.

Definition 3.4 A BDM-Algebra $\boldsymbol{X}$ is called 0 -commutative if $r \ominus (0 \ominus s) = s \ominus (0 \ominus r)$ for any $r, s \in \mathfrak{R}$.

Lemma 3.1 Let $\not{b}$e a BDM-Algebra. Then, $x \ominus s = \ominus(0 \ominus (0 \ominus s))$ for each $f, s \in \mathfrak{R}$.

Proposition 3.1 Let ↖ be a 0 -commutative BDM-Algebra. Then, $(0\ominus)\ominus (0 \ominus s) = s \ominus r$ for any $x, s \in \Psi$.

Proof: Since $\mathbb{R}$ is a 0 -commutative, then $x \ominus (0 \ominus s) = s \ominus (0\ominus f)$for any $x, s \in \mathbb{R}$. By Lemma 3.1, we get $s \ominus r = s \ominus (0 \ominus (0 \ominus x)) = (0 \ominus r) \ominus (0 \ominus s)$. Therefore, as required.

Definition 3.5 Let Ψ and $\mathcal{X}'$ be two BDM-Algebras. A mapping $F : \Psi \to X'$ is called BDM-homomorphism, if $F(x\ominus s) = F(r) \ominus' F(s)$ for any $x, s \in \mathbb{Z}$.

Remark 3.3 Let $F : \Psi \to \mathbb{K}'$ be a BDM-homomorphism. If F is a surjective mapping, then it's called BDM-epimorphism. Furthermore, if F is one-to-one, then F is called BDM-monomorphism and if F is a bijection, then F is called BDM-isomorphism.

Proposition 3.2 Let Ψ and Ψ' be two BDM-Algebras. Moreover, let $F : \Psi \to \Psi'$ be a BDM-homomorphism. Then,

1.If Ψ is a BDM-sub-algebra of $\mathcal{R}$, then $F(\mathcal{K})$ is a BDM-sub-algebra of $\mathcal{Z}'$.

2. If Ψ is a BDM-sub-algebra of ⁄, then $F^{-1}()$ is a BDM-sub-algebra of ⁄

Proof: (1) Let $F() = \{f \in \mathcal{Z} : F() \in \mathcal{X}'\}$. Since $\mathcal{W}$ is a BDM-sub-algebra of $\mathcal{Z}$, then by Definition 3.2, $x \ominus s \in \mathcal{W}$ for any $r, s \in \mathcal{H}$. Now, $F(f \ominus s) \in F(\mathcal{K}) = F(f)\ominus F(s) \in F()$. Thus, $F()$ is a BDM-sub-algebra of ⁄. (2) Let $F^{-1}() = \{f \in . f(r) \in W$. That is mean $r \in F^{-1}() \Leftrightarrow F() \in W$. Now, let $r, s \in F^{-1}()$ then $F(), F(s) \in W$. But $\mathcal{W}$ is a BDM-sub-algebra of $\mathcal{K}'$. This gives us $F(F) \ominus F(s) \in \Psi \Rightarrow F(r \ominus s) \in \mathcal{W}$ which gives that $r \ominus s \in F^{-1}(\mathcal{X})$. Therefore, $F^{-1}()$ is a BDMsub-algebra of $\mathfrak{R}$.

Proposition 3.3 Let $F : \Psi \to \mathcal{R}'$ and $F^{\circ} : \mathbb{R}' \to \mathcal{R}''$ be two BDM-homomorphisms. Then, $(F \circ F^{\circ}) : \Psi \to \mathbb{R}''$ is a BDM-homomorphism.

Proof: Since F is a BDM-homomorphism, then by Definition 3.5, $F(f \ominus s) = F(r) \ominus' F(s)$ for any $x, s \in \Psi$ and similarly for a BDM-homomorphism F°. Now, $(F \circ F^{\circ})(r \ominus s) = F(F^{\circ}(r \ominus s)) = F(F^{\circ}(r) \ominus F^{\circ}(s)) = F(F^{\circ}(r)) \ominus F(F^{\circ}(s)) = (F\circ F^{\circ})(F) \ominus (F \circ F^{\circ})(s)$. Therefore, $(F \circ F^{\circ})$ is a BDM-homomorphism.

Definition 3.6 Let $F : \Psi \to \mathbb{R}'$ be a BDM-homomorphism. Then, kernel of F is defined as $\operatorname{Ker} F = \{f \in \mathfrak{R} : F(F) = 0\}$.

Proposition 3.4 Let $f : \mathbb{R} \to \mathbb{R}'$ be a BDM-homomorphism. Then, $KerF$ is a BDMsub-algebra of $\mathfrak{R}$.

Proof: Let $, s \in \operatorname{Ker} F$ to show that $x \ominus s \in \operatorname{Ker} F$. Since $r, s \in \operatorname{Ker} F$ then $F(r) = 0$ and $F(s) = 0$. Now, $F(r \ominus s) = F(f) \ominus F(s) = 0 \ominus 0 = 0$. Thus, $f \ominus s \in \operatorname{Ker} F$. Therefore, Ker F is a BDM-sub-algebra of Ψ.

Definition 3.7 Let $\mathfrak{R}$ be a BDM-algebra and $\Phi \neq\subseteq \Psi$. Then, $\mathcal{W}$ is said to be an ideal of Ψ, if it obeys the following points.

1. $0 \in \mathcal{W}$,

2. If $r \ominus s \in Ж$ and $s \in Ж \Longrightarrow r \in Ж$ for any $r, s \in Ж$.

Definition 3.8 An ideal $\mathcal{W}$ of $\mathcal{X}$ is said to be normal of $\mathcal{R}$, if $(f \ominus s) \in \mathcal{W}(t\ominus) \ominus(t \ominus s) \in \mathcal{H}$ for any $r, s, t \in \mathcal{H}$.

Definition 3.9 Let $\Phi \neq Ж \subseteq Ѫ$, then Ж is called normal of Ѫ, if $(r \ominus t) \ominus (s \ominus t) \in$ for each $(r \ominus s), (t \ominus t) \in$

Corollary 3.1 Any normal subset of $\mathbb{X}$ is a BDM-sub-algebra of $\mathbb{Z}$ but the convers is not necessary true.

Proof: Let Ψ be a normal subset of $\mathcal{X}$, then by Definition 3.9, $(r \ominus t) \ominus (s \ominus t) \in$ for each $(r \ominus s), (t \ominus t) \in$. Therefore, by Definition 3.2, we get Ψ is a BDMsub-algebra of Ψ.

Example 3.6 A subset $\mathcal{W} = \{0, 3\}$ is a BDM-sub-algebra of $\mathcal{X}$ but it's not normal, because $0 \ominus 3, 1 \ominus 2 \in \{0, 3\}$ but $(0 \ominus 1) \ominus (3 \ominus 2) = 2 \notin \{0, 3\}$.

Proposition 3.5 Let $F : \Psi \to \mathfrak{R}'$ be a BDM-homomorphism. Then, $KerF$ is an ideal of .

Proof: Clearly that $0 \in \operatorname{Ker} F$ since $F(0) = 0$. Let $r \ominus s \in \operatorname{Ker} F$ and $s \in \operatorname{Ker} F$, then $F(s) = 0$ and $F(x \ominus s) = F(F) \ominus' F(s) = F(F) \ominus' 0 = 0$. From the other side, $F() = 0$ which give that $F \in \operatorname{Ker} F$. Therefore, $\operatorname{Ker} F$ is an ideal of $\mathfrak{R}$.

Proposition 3.6 Let $F : \mathbb{R} \to \mathbb{K}'$ be a BDM-homomorphism. Moreover, let $F : \mathbb{R} \to \mathcal{K}'$ be an epimorphism. If $\mathcal{W}$ is an ideal of $\mathcal{R}$, then $F(\mathbb{K})$ is an ideal of $\mathcal{K}'$.

Proof: Since $\mathcal{W}$ be an ideal of $\mathcal{X}$, then by Definition 3.7, we have $0 \in \mathcal{W}$ and $r \ominus s \in$ with $s \in \mathcal{W}$ implies $f \in \mathcal{W}$ for any , $s \in \mathcal{H}$. Since $F : \Psi \to \mathbb{R}'$ be an epimorphism, then $F : \mathbb{R} \to \mathbb{K}'$ is an onto. This will gives that $F(0) \in F(\mathcal{K})$ and $F(F) \in F(\mathcal{K})$. From the assumption we have $F :\twoheadrightarrow K'$ is a BDM-homomorphism and this gives us $F(f \ominus s) = F(F) \ominus' F(s)$. Again, since $F :\twoheadrightarrow \mathfrak{R}'$ be an epimorphism, then for any $t, t \in \mathcal{R}'$ there exist $f, s \in W$ such that $F(F) = t$ and $F(s) = t$. Now, let $t, t \in F() \subseteq \mathbb{K}'$, then $t \ominus t = F(\mathfrak{F}) \ominus F(s) = F(f \ominus s) \in F(\mathcal{K}) \subseteq \mathbb{K}'$. Therefore, $F(\mathbb{K})$ is an ideal of $\mathbb{X}'$.

Remark 3.4 Let $\mathbb{X}$ be a BDM-Algebras and let $\mathcal{W}$ be a normal ideal of Ψ. We define $\sim$ for any , $s \in \mathbb{R}$ by $\widetilde{\mathcal{K}}s \Leftrightarrow r \ominus s \in \mathcal{H}$.

Proposition 3.7 Let $\mathcal{K}$ be a BDM-Algebras and let $\mathcal{W}$ be a normal ideal of $\mathcal{X}$. Then, $\sim$ is an equivalence relation of $\mathbb{K}$.

Proof: Since every ideal is a normal ideal, then $0 \in \mathcal{W}$ and by Definition 3.1, we have $\mathbf{r} \ominus \mathbf{r} = \mathbf{r} \in \mathbb{X}$. That is mean $\mathbf{r}\widetilde{\mathcal{K}}\mathrm{r} \Leftrightarrow \mathbf{r} \ominus \mathbf{r} \in \mathcal{W}$ for any $r \in \mathcal{X}$ which gives that $\sim$ is a reflexive. To show that $\sim$ is a symmetric, let $x\widetilde{K}s$ for any , $s \in \mathbb{R}$ to show that $s\widetilde{\mathcal{K}}r$. Since $\widetilde{\mathcal{K}}s$ then, $r \ominus s \in \mathcal{H}$. But $\mathcal{W}$ is a normal. Thus, $(t \ominus r) \ominus (t \ominus s) \in \mathcal{W}$ for any $r, s, t \in \mathcal{H}$. By Definition 3.1, we get $(t \ominus r) \ominus (t \ominus s) = s \ominus r \in \mathrm{R}$. Since $\mathcal{W}$ be an ideal, then $s \ominus r \in \mathcal{H}$. Thus, $s\widetilde{\mathcal{K}}r$. Hence, $\sim$ is a symmetric. Finally, let $\mathrm{r}\widetilde{\mathcal{K}}\mathrm{s}$ and $\mathrm{s}\widetilde{\mathcal{W}}\mathrm{t}$ to prove that $\mathrm{r}\widetilde{\mathcal{K}}\mathrm{t}$. Since $\mathrm{r}\widetilde{\mathcal{K}}\mathrm{s}$ and $\mathrm{s}\widetilde{\mathcal{K}}\mathrm{t}$, then $\mathrm{r} \ominus \mathrm{s} \in \mathcal{W}$ and $\mathrm{s} \ominus t \in \mathcal{H}$. Since $\mathcal{W}$ is a normal ideal, then $(t\ominus) \ominus (t \ominus s) \in \mathcal{W}$ and $(f \ominus s) \ominus (f \ominus t) \in \mathrm{W}$. By Definition 3.1, $(t \ominus r) \ominus (t \ominus s) = (s \ominus r) \in \mathrm{W}$ and we have $(t \ominus s) \in \mathrm{W}$. Thus, we get $(s \ominus r), (t \ominus s) \in \mathcal{H}$. Since $\sim$ is symmetric, then $(s \ominus t), (s\ominus) \in$. By Definition 3.1, $(s \ominus \xi) \ominus (s \ominus r) = (r \ominus \xi) \in$ which gives that $\widetilde{\mathcal{K}}t$. Thus, $\sim$ is a transitive. Therefore, $\sim$ is an equivalence relation of Ψ.

Remark 3.5 Let $\mathfrak{X}$ be a BDM-Algebras and let Ψ be a normal ideal of Ψ. The congruent class that containing $f \in \mathbb{Z}$ is defined as $[f]_{\mathcal{X}} = \{s \in \mathbb{R} : \mathfrak{f} \sim s\}$. Therefore, $f \sim \mathrm{s} \Leftrightarrow [\mathrm{f}]_{\mathcal{K}} = [\mathrm{s}]_{\mathcal{W}}$. Furthermore, the set of all equivalence classes of $\mathcal{K}$ is denoted by $\mathfrak{R}/\mathcal{K} = \{[r]_{\mathcal{W}} :\in \mathbb{R}\}$.

Theorem 3.1 Let Ψ be a BDM-Algebras and let Ψ be a normal ideal of Ψ. Then, $/\!\!/$ is a BDM-Algebras.

Proof: By Proposition 3.7, is an equivalence relation of $\boldsymbol{X}$. Moreover, let $[\mathrm{f}]_{\mathcal{K}} \ominus [s]_{\mathcal{H}} = [\mathrm{r} \ominus \mathrm{s}]_{\mathcal{W}}$, then $\ominus$ is well-defined. Since, if $t\widetilde{W}r$ and $t\widetilde{W}s$ then $t \ominus r \in \mathcal{H}$ and $\mathrm{t} \ominus s \in \mathrm{W}$. By normality of $\mathcal{W}(\mathrm{t}\ominus\mathrm{t})\ominus(\mathrm{t}\ominus\mathfrak{f}) \in$ and $(\mathrm{f}\ominus\mathrm{t})\ominus(\mathfrak{f}\ominus\mathrm{s}) \in$. The symmetric of $\sim$ gives that $(t \ominus r)\widetilde{W}(f \ominus t)$ and the transitivity gives $(t \ominus \mathrm{t})\ \widetilde{\Psi}(r \ominus s)$ which implies $(t \ominus t)\widetilde{\mathcal{K}}(\mathrm{r} \ominus s)$. Since $\sim$ is a reflexive relation, then for any $r \in \mathbb{R}$ we have $r\widetilde{T}r$ and $r \ominus r \in \mathcal{K}$. Now, $[r]_{\mathcal{K}} \ominus [r]_{\mathcal{K}} = [r \ominus r]_{\mathcal{K}} = \{r \in: r\widetilde{W}r\} = \{r \in \mathfrak{R} : r \ominus r \in\} = \{r \in \mathfrak{R} : r \in\} = [r]_{\mathcal{W}}$. Next, $[\mathrm{f}]_{\mathcal{W}} \ominus [0]_{\mathcal{W}} = [\mathrm{r} \ominus 0]_{\mathcal{W}} = \{\mathrm{f} \in : \mathfrak{F}\widetilde{\mathscr{W}}0\} = \{\mathrm{f} \in : \mathrm{f} \ominus 0 \in \mathcal{W}\} = \{\mathrm{f} \in : \mathrm{f} \in \} = [\mathrm{r}]_{\mathcal{W}}$. Finally, if $[\mathrm{f}]_{\mathcal{W}}, [\mathrm{s}]_{\mathcal{W}}, [\mathrm{t}]_{\mathcal{W}} \in \mathbb{R}/\mathcal{W}$, then $([\mathrm{F}]_{\mathcal{K}} \ominus [\mathrm{s}]_{\mathcal{W}}) \ominus ([\mathrm{F}]_{\mathcal{K}} \ominus [t]_{\mathcal{W}}) = [\mathrm{f}\ominus\mathrm{s}]_{\mathcal{W}} \ominus [\mathrm{f}\ominus\mathrm{t}]_{\mathcal{W}} = [(\mathrm{f}\ominus\mathrm{s})\ominus(\mathrm{f}\ominus\mathrm{t})]_{\mathcal{W}} = [\mathrm{t}\ominus\mathrm{s}]_{\mathcal{W}} = [\mathrm{t}]_{\mathcal{W}} \ominus [\mathrm{s}]_{\mathcal{W}}$. Therefore, $\mathfrak{R}/$ is a BDM-Algebras.

Definition 3.10 Let $(\Psi, \ominus, 0)$ and $(\Psi', \ominus', 0')$ be two BDM-Algebras. We define the direct product of $\mathbb{X}$ and $\mathbb{X}'$ by $(\mathcal{K} \times \mathbb{X}', \Box, (0, 0'))$ with the binary operation given as $(\mathrm{f}_1, \mathrm{s}_1) \Box (\mathrm{F}_2, \mathrm{s}_2) = (\mathrm{f}_1 \ominus \mathrm{F}_2, \mathrm{s}_1 \ominus \mathrm{s}_2)$ where $\mathscr{K} \times \mathbb{K}' =$
$\{(\mathrm{F}, \mathrm{s}) : \mathrm{f} \in \mathbb{R}, \mathrm{s} \in \mathbb{K}'\}$.

Theorem 3.2 A systems $(\mathcal{R}, \ominus, 0)$ and $(\mathbb{K}', \ominus, 0')$ are BDM-Algebras iff $(\mathcal{K} \times \mathbb{R}', \Box\ , (0, 0'))$ is a BDM-Algebra.

Proof: Let $(\Psi, \ominus, 0)$ and $(\Psi', \ominus, 0')$ are BDM-Algebras. From the assumption we have $r \ominus r = r$ for any $r \in \mathbb{K}$ and $s \ominus s = s$ for any $s \in \mathbb{K}'$. Thus, $(r, s)\Box(r, s) = (f \ominus r, s \in s) = (r, s) \in \mathbb{R} \times \mathbb{R}'$. This proves condition one. Also from the assumption we have $x \ominus 0 = x$ for any $r \in \mathbb{Z}$ and $s \ominus 0' = s$ for any $s \in \mathbb{K}'$. So that $(f, s)[\ (0, 0') = (f \ominus 0, s \ominus 0') = (f, s) \in \mathbb{R} \times \mathbb{R}'$. This proves condition two. Finally, let $(\mathrm{r} \ominus s) \ominus (\mathrm{F} \ominus \mathrm{t}) = (t \ominus s)$ and $(\mathrm{F}' \ominus \mathrm{s}') \ominus (\mathrm{F}' \ominus \mathrm{t}') = (\mathrm{t}' \ominus \mathrm{s}')$. Thus, $[(\mathrm{f}\ominus \mathrm{F}') \Box (s \ominus \mathrm{s}')] \ominus [(\mathrm{f} \ominus \mathrm{F}') \Box (t \ominus \mathrm{t}')] = ((\mathrm{F} \ominus \mathrm{s}), (\mathrm{F}' \ominus \mathrm{s}')) \ominus ((\mathrm{f} \ominus \mathrm{t}), (\mathrm{F}' \ominus\ \mathrm{t}')) = ((\mathrm{r} \ominus \mathrm{s}) \ominus (\mathrm{x} \ominus \mathrm{t})), ((\mathrm{k}' \ominus \mathrm{s}') \ominus (\mathrm{F}' \ominus \mathrm{t}')) = (\mathrm{t} \ominus \mathrm{s}, \mathrm{t}' \ominus \mathrm{s}') \in \mathbb{R} \times \mathbb{K}'$. Hence, condition three is proved. Therefore, $(\mathbb{R} \times \mathbb{K}', \Box, (0, 0'))$ is a BDM-Algebra. Conversely, let $(\mathbb{R} \times \mathbb{R}', \Box, (0, 0'))$ is a BDM-Algebra, then, $(f, s)\Box(f, s) = (f \ominus \mathrm{r}, \mathrm{s} \ominus \mathrm{s}) = (\mathrm{r}, \mathrm{s})$ and this gives that $\mathrm{r} \ominus \mathrm{r} = \mathrm{r}$ and $\mathrm{s} \ominus \mathrm{s} = \mathrm{s}$ for any $\mathrm{r} \in \boldsymbol{R}, \mathrm{s} \in \Psi'$. Moreover, $(f, s)\Box(0, 0') = (f \ominus 0, s \ominus 0') = (f, s)$ which gives $\mathrm{r} \ominus 0 = f$ and $s \ominus 0' = s$ for any $f \in \mathbb{R}, s \in \mathbb{R}'$. Ultimately, $(t \ominus s, t' \ominus s') \in \mathbb{Z} \times \mathbb{R}'$. That is mean we have $((\mathrm{F} \ominus s) \ominus (\mathrm{F} \ominus \mathrm{t})), ((\mathrm{F}' \ominus \mathrm{s}') \ominus (\mathrm{F}' \ominus \mathrm{t}')) \in \mathbb{R} \times \mathbb{R}'$. This implies that $(f \ominus s) \ominus (x \ominus t) = (t \ominus s)$ for any $x, s, t \in \mathbb{R}$ and $(f' \ominus s') \ominus (F' \ominus t') = (t' \ominus s')$ for any $\mathbf{f}', s', t' \in \mathbb{R}'$. Therefore, this completed the proof.

Definition 3.11 Let $\{(\mathfrak{R}_i, \ominus, 0_i), \mathrm{i} = 1, \cdots, \mathfrak{n}\}$ be a collection of BDM-Algebras. We define the direct product of Ψ_{i} where $\mathrm{i} = 1, \cdots, \mathfrak{n}$ by $(\prod_{i=1}^{\mathfrak{n}} \Psi_{\mathrm{i}}, \Box, (0_1, \cdots, 0_{\mathfrak{n}}))$ such that $\prod_{i=1}^{\mathfrak{n}} \Psi_{\mathrm{i}} = \Psi_1 \times \cdots \times \nearrow_{\mathfrak{n}} = \{(\mathfrak{r}_1, \cdots, \mathfrak{x}_{\mathfrak{n}}) : \mathfrak{x}_{\mathrm{i}} \in \Psi_{\mathrm{i}}, \forall \mathrm{i}\}$. Furthermore, $(\mathfrak{r}_1, \cdots, \mathfrak{r}_{\mathfrak{n}}) \Box (\mathrm{s}_1, \cdots, \mathrm{s}_{\mathfrak{n}}) = (\mathfrak{r}_1 \ominus \mathrm{s}_1, \cdots, \mathfrak{r}_{\mathfrak{n}} \ominus \mathrm{s}_{\mathfrak{n}})$.

Corollary 3.2 Let $\{(\mathfrak{R}_i, \ominus, 0_i), \mathrm{i} = 1, \cdots, \mathfrak{n}\}$ be a collection of BDM-Algebras. Then, $\left(\prod_{i=1}^{\mathfrak{n}} \Psi_i, \Box, (0_1, \cdots, 0_{\mathfrak{n}})\right)$ is a BDM-Algebras.

Theorem 3.3 Let $\{(\mathfrak{R}_i, \ominus, 0_i), \mathrm{i} = 1, \cdots, \mathfrak{n}\}$ be a collection of BDM-Algebras. Then, $\nearrow_{\mathrm{i}}$ is 0 -commutative iff $\prod_{i=1}^{n} \mathscr{R}_i$ is 0 -commutative.

Proof: Suppose that Ψ_i is 0 -commutative to show that $\prod_{i=1}^{\mathfrak{n}} \mathscr{K}_i$ is 0-commutative. Since $\boldsymbol{K}_{\mathrm{i}}$ is 0 -commutative for each i, then we have $r_i \ominus (0_{\mathrm{i}} \ominus s_{\mathrm{i}}) = s_i \ominus (0_{\mathrm{i}} \ominus r_{\mathrm{i}})$ for each $r_{\mathrm{i}}, s_i \in K_i$. Thereafter, $(r_1, \cdots, \mathfrak{r}_{\mathfrak{n}}) \ominus ((0_1, \cdots, 0_{\mathfrak{n}}) \ominus (s_1, \cdots, s_{\mathfrak{n}})) = (\mathfrak{r}_1, \cdots, \mathfrak{x}_{\mathfrak{n}}) \ominus (0_1 \ominus s_1, \cdots, 0_{\mathfrak{n}} \ominus s_{\mathfrak{n}}) = \mathfrak{r}_1 \ominus (0_1 \ominus s_1), \cdots, \mathfrak{r}_{\mathfrak{n}} \ominus (0_{\mathfrak{n}} \ominus s_{\mathfrak{n}}) = s_{\mathrm{i}} \ominus (0_{\mathrm{i}} \ominus \mathfrak{r}_{\mathrm{i}}) = (s_1, \cdots, s_{\mathfrak{n}}) \ominus ((0_1 \ominus \mathfrak{r}_1, \cdots, 0_{\mathfrak{n}} \ominus \mathfrak{r}_{\mathfrak{n}})) =$

$(s_1,\cdots,s_n)\ominus((0_1,\cdots,0_n)\ominus\ (r_1,\cdots,r_n))$ for any $(r_1,\cdots,r_n),(s_1,\cdots,s_n)\in\prod_{i=1}^{n} x_i$. Therefore, $\prod_{i=1}^{n} x_i$ is a 0 -commutative. Conversely, let $\prod_{i=1}^{n} x_i$ is a 0-commutative, then we have $(r_1,\cdots,r_n)\ominus((0_1,\cdots,0_n)\ominus(s_1,\cdots,s_n)) = (s_1,\cdots,s_n)\ominus((0_1,\cdots,0_n)\ominus(r_1,\cdots,r_n))$ for any $(r_1,\cdots,r_n),(s_1,\cdots,s_n)\in\prod_{i=1}^{n} x_i$. So that, $r_i\ominus(0_i\ominus s_i) = (r_1\ominus(0_1\ominus s_1),\cdots,(r_n\ominus(0_n\ominus s_n)) = (r_1,\cdots,r_n)\ominus((0_1,\cdots,0_n)\ominus(s_1,\cdots,s_n)) = (s_1,\cdots,s_n)\ominus((0_1,\cdots,0_n)\ominus(r_1,\cdots,r_n)) = (s_1\ominus(0_1\ominus r_1),\cdots,(s_n\ominus(0_n\ominus\ r_n)) = s_i\ominus(0_i\ominus x_i)$ for each $r_i, s_i\in_{i}\!\!/$ Therefore, $\nearrow_i$ is 0-commutative.

Remark 3.6 Let $\{(\mathbb{K}_i,\ominus,0_i), i=1,\cdots,n\}$ and $\left\{\left(\mathcal{R}_i',\ominus,0_i\right), i=1,\cdots,n\right\}$ be two collections of BDM-Algebras such that $\mathbb{K}_i\cong\mathbb{K}_i'$ for each $i=1,\cdots,n$. Then, $\prod_{i=1}^{n} {}_i\!\!\not\cong\prod_{i=1}^{n}\mathfrak{R}_i'$.

Theorem 2.4 Let $\{(\mathfrak{R}_i,\ominus,0_i), i=1,\cdots,n\}$ be a collection of BDM-Algebras with $\mathcal{K}_i$ is a normal ideal of Ψ_i for each $i=1,\cdots,n$. Then, $\prod_{i=1}^{n}\mathcal{K}_i$ is a normal ideal of $\prod_{i=1}^{n}\Psi_i$. Furthermore, $\prod_{i=1}^{n} K_i/\prod_{i=1}^{n}\mathcal{K}_i\cong\prod_{i=1}^{n}(\aleph_i/\mathcal{K}_i)$.

Proof: Since $\mathcal{K}_i$ is a normal ideal of $\nearrow_i$ for each $i=1,\cdots,n$, then $0_i\in\mathcal{K}_i$ which gives that $\prod_{i=1}^{n}\mathcal{K}_i\neq\Phi$. From this we get $(0_1,\cdots,0_n)\in\prod_{i=1}^{n}\mathcal{K}_i$. Since Ψ_i is a BDMAlgebra for each $i=1,\cdots,n$, then $r_i, s_i\in\Psi_i$ which implies $(r_1,\cdots,r_n),(s_1,\cdots,s_n)\in\prod_{i=1}^{n}\aleph_i$. Furthermore, $\mathcal{K}_i$ is a normal ideal of $\aleph_i$ for each $i=1,\cdots,n$, then $r_i\ominus s_i$ and s_i belong to $\mathcal{W}_i$. This gives that $(r_1\ominus s_1,\cdots,r_n\ominus s_n),(s_1,\cdots,s_n)\in\prod_{i=1}^{n}\Psi_i$ which gives us $(r_1,\cdots,r_n)\in\prod_{i=1}^{n}\mathcal{K}_i$. Thus, $\prod_{i=1}^{n}\mathcal{K}_i$ is an ideal of $\prod_{i=1}^{n}\mathfrak{R}_i$. Now, let $t_i\in\Psi_i$ for each $i=1,\cdots,n$, then $(t_1,\cdots,t_n)\in\prod_{i=1}^{n}\mathbb{K}_i$. So that $((t_1,\cdots,t_n)\,\square\,(f_1,\cdots,f_n))\,\square\,((t_1,\cdots,t_n)\,\square\,(s_1,\cdots,s_n)) = (t_1\ominus r_1,\cdots,t_n\ominus r_n)\,\square\,(t_1\ominus s_1,\cdots,t_n\ominus s_n) = ((t_1\ominus x_1)\ominus(t_1\ominus s_1),\cdots,(t_n\ominus x_n)\ominus(t_n\ominus s_n))\in\prod_{i=1}^{n}\Psi_i$.

Therefore, $\prod_{i=1}^{n}\mathcal{K}_i$ is a normal ideal of $\prod_{i=1}^{n}\boldsymbol{K}_i$. Next, to prove that

$$\prod_{i=1}^{n}\text{Ѫ}_i/\prod_{i=1}^{n}\text{Ж}_i\ \cong\prod_{i=1}^{n}(\text{Ѫ}_i/\text{Ж}_i).\ \text{Let } F:\prod_{i=1}^{n}\text{Ѫ}_i/\prod_{i=1}^{n}\text{Ж}_i\ \longrightarrow\prod_{i=1}^{n}(\text{Ѫ}_i/\text{Ж}_i)\ \text{such}$$

$$\text{that } F(r_i/\text{Ж}_i) = r_1/\text{Ж}_1,\cdots,r_n/\text{Ж}_n\ \text{for each } r_1/\text{Ж}_1,\cdots,r_n/\text{Ж}_n\in\prod_{i=1}^{n}\text{Ѫ}_i/\prod_{i=1}^{n}\text{Ж}_i\ .$$

To show that F is well-defined, let $r_i/\text{Ж}_i, s_i/\text{Ж}_i\in\prod_{i=1}^{n}\text{Ѫ}_i/\prod_{i=1}^{n}\text{Ж}_i$ with $r_i/\aleph_i = s_i/\mathcal{W}_i$ which gives that $(r_1/\aleph_1,\cdots,r_n/_n) = (s_1/_1,\cdots,s_n/_n)$. That is mean $(r_1,\cdots,r_n)\,\widetilde{K}\,(s_1,\cdots,s_n)$ which implies that $(r_1,\cdots,r_n)\,\square\,(s_1,\cdots,s_n) = (r_1\ominus\ s_1,\cdots,r_n\ominus s_n)\in\prod_{i=1}^{n}\mathscr{K}_i$. From this we get $r_i\ominus s_i\in\mathscr{W}_i$ with $r_i\widetilde{\mathscr{K}}s_i\in\mathcal{K}_i$ and $r_i/\text{Ж}_i = s_i/\text{Ж}_i$. Thereby, $F(r_1,\cdots,r_n/\prod_{i=1}^{n}\text{Ж}_i) = r_1/\text{Ж}_1,\cdots,r_n/\text{Ж}_n = s_1/\text{Ӿ}_1,\cdots,s_n/\text{Ӿ}_n = F(s_1,\cdots,s_n/\prod_{i=1}^{n}\mathcal{W}_i)$. Therefore, F is well-defined. Now, let $(r_1,\cdots,r_n)/\prod_{i=1}^{n}\Psi_i,(s_1,\cdots,s_n)/\prod_{i=1}^{n}\Psi_i\in\prod_{i=1}^{n}\bigcap_i/\prod_{i=1}^{n}\Psi_i$, then $F((r_1,\cdots,r_n)\sqcup(s_1,\cdots,s_n)/\prod_{i=1}^{n}\mathcal{W}_i\Big) = F((r_1\ominus s_1),\cdots,(r_n\ominus s_n)/\prod_{i=1}^{n}\Psi_i) = ((r_1\ominus s_1)/\text{Ӿ}_1,\cdots,(r_n\ominus s_n)/_n) = (r_1/\Psi_1,\cdots,r_n/\aleph_n)\,\square\,(s_1/_1,\cdots,s_n/\Psi_n) = F((r_1,\cdots,r_n)/\prod_{i=1}^{n}\Psi_i)\,\square F\left((s_1,\cdots,s_n)/\prod_{i=1}^{n}\Psi_i\right)$. Hence, F is a BDM-homomorphism. Next, if $F\left((r_1,\cdots,r_n)/\prod_{i=1}^{n}\mathcal{F}_i\right) = F\left((s_1,\cdots,s_n)/\prod_{i=1}^{n}\mathcal{W}_i\right)$ then $(r_1,\cdots,r_n)/\prod_{i=1}^{n}\Psi_i\Big) = F\left((r_1,\cdots,r_n)/\prod_{i=1}^{n}\mathcal{W}_i\right) = F\left((s_1,\cdots,s_n)/\prod_{i=1}^{n}\mathcal{W}_i\right) = (s_1,\cdots,s_n)/\prod_{i=1}^{n}\text{Ж}_i\Big)$. Since $r_i/\text{Ж}_i = s_i/\text{Ж}_i$ for each $i=1,\cdots,n$, then $r_i\widetilde{\text{Ж}}\,s_i\in\text{Ж}_i$ and $r_i\ominus s_i\in\mathcal{K}_i$. Thereafter, $(r_1,\cdots,r_n)\,\square\,(s_1,\cdots,s_n) = (r_1\ominus s_1,\cdots,r_n\ominus s_n)\in\prod_{i=1}^{n}\mathcal{W}_i$ which gives that $(r_1,\cdots,r_n)\,\widetilde{\Psi}\,(s_1,\cdots,s_n)$ and from this we get $(r_1,\cdots,r_n)/\prod_{i=1}^{n}\mathcal{F}_i\Big) = (s_1,\cdots,s_n)/\prod_{i=1}^{n}\mathcal{W}_i\Big)$. Thereafter, F is $(1-1)$. Finally, let $(r_1/\mathcal{K}_1),\cdots,r_n/\Psi_n)\in\prod_{i=1}^{n}(\mathfrak{R}_i/\Psi_i)$, then $r_i\in\mathbb{K}_i$ for each $i=1,\cdots,n$. That is mean $(r_1/_1),\cdots,r_n/\aleph_n) = F(r_1,\cdots,r_n)/\prod_{i=1}^{n}\mathcal{W}_i\Big)$ where $(r_1,\cdots,r_n)/\prod_{i=1}^{n}\mathcal{W}_i\in\prod_{i=1}^{n}K_i/\prod_{i=1}^{n}\mathscr{K}_i$ which is proved that F is an onto. Therefore, $\prod_{i=1}^{n}\mathcal{K}_i/\prod_{i=1}^{n}\mathcal{K}_i\cong\prod_{i=1}^{n}(\aleph_i/\aleph_i)$.

4. CONCLUSION

As a result of this study, the concept of BDM-Algebras has been introduced with some of its properties. We proved that every normal sub-algebra of a BDM-Algebras is a BDM-sub-algebra but the convers is not necessary true. Also we showed that the image (pre-image) of a BDM-sub-algebra of a BDM-Algebra is a BDM-sub-algebra. Then, the composition of two BDM-homomorphisms is a BDM-homomophism and the kernel of a BDM-homomorphism is a BDM-sub-algebra of a BDM-Algebra. Furthermore, we proved that the kernel of a BDM-homomorphism is an ideal. Ultimately, this study proved that the direct product of two BDM-Algebras is a BDM-Algebra and the direct product of a finite family of a BDM-Algebras is a 0 -commutative if and only if each one of them is a 0 -commutative.

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CONFLICTS OF INTEREST

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