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The Analytical Methods of Volterra Integral Equations of The Second Kind

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ABSTRACT: This paper discussed the analytic methods to solve Second order Volterra integral equations form using different methods. The domain decomposition method is the first technique. Depending on the hypothesis, the solution is by sequence. The second method is the successive approximation technique, which is used Picard iteration method. The third method used Laplace transformation. The modified decomposition technique is used as the fourth method. Depending on the Taylor series, the fifth method is called the series method. The last method solves the VIE using the functional correction technique called the variational iteration method. We introduce some examples to illustrate these methods.

Keywords: The Domain Decomposition Method, the Successive Approximation Technique, The Laplace Transformation Method, the Modified Decomposition Technique, The Series Solution Technique, and The Variational Iteration Technique.

1. INTRODUCTION

The Volterra integral equations and methods for solving them are covered in this chapter. Vito Volterra (1860–1940), Ivar Fredholm (1860–1927), David Hilbert (186–1943), and Erhard Schmidt are the top researchers in the field of integral equations (1876-1959). Volterra was the primary to acknowledge the significance of the hypothesis and conduct a thorough investigation into it. We'll discuss the second nonhomogeneous Volterra integral equation of the following form. We'll start by examining the second form of the Volterra integral equation is stated by:

$$\omega(y)\ \mu(y) = \phi(y) + \lambda \int_{\alpha}^{y} k(y,t)\, \mu(t)\, dt. \tag{1}$$

The desired unidentified function, μ(y), is the integral indication inside and outside. Given are real-valued functions for the function ϕ(y),K(y,t), and the kernel. With y as the parameter. Here is an explanation of both contemporary and traditional methods. [1, 2] An integral equation was initially created in 1825 by the Italian mathematician Abel on the well-known tautochrone problem. [3] The problem is determining the curve's length along which a heavy particle can slide without resistance and reach its lowest point. [4, 5] Numerous applications in science and engineering use integral differential equations. Initial value issues with predetermined prime numbers can be transformed into Volterra integral equations. [6] However, from boundary value problems with well-known boundary conditions, Fredholm's integral equations and Fredholm's differential integral equations can be derived. It is significant to note that transformations are commonly used in the literature to get from simple value problems to the integrality of Volterra's equations and initial value problems to Volterra's integral equations. However, it is uncommon to translate boundary value issues into analogous uniform integral equations and the other way around. [7]

2. ANALYTICAL METHODS FOR SOLVING VIES

We will start by looking at the second category of Volterra integral equations offered by:

$$\mu(y) = \phi(y) + \lambda \int_0^y K(y,t)\,\mu(t)\,dt. \tag{2}$$

There are examples of the undefinable function μ(t) inside and outside the integral sign. Both the kernel K(y,t) and the function ϕ(y), where y is a parameter, become available as real-valued functions [6]. The unconventional and conventional methods that will be employed are described in the following sections [4].

2.1 DOMAIN DECOMPOSITION TECHNIQUE

George A. Domain introduced the A domain decomposition method (ADM), which he developed. Any unknown function, $\mu(y)$, is divided into an infinite number of components, each of which is defined by the A domain decomposition approach to decompose the series: [6]

$$\mu(y) = \sum_{m=0}^{\infty} \mu_m(y). \tag{3}$$

Or equivalently $\mu(y) = \mu_0(y) + \mu_1(y) + \mu_2(y) + \dots$.

In this case, a recursive process must be used to find the components $\mu_m(y)$, $m \geq 0$. The decomposition method focuses on locating each component separately; we enter (3) into the Volterra integral equation (2) to get: [8]

$$\sum_{m=0}^{\infty} \mu_m(y) = \phi(y) + \int_{\alpha}^{y} k(y,\ t)\left(\sum_{m=0}^{\infty} \mu_m(t)\right) dt. \tag{4}$$

All terms that are not covered by the integral sign refer to the $\mu_0(y)$ zeroth component. As a result, by establishing the recurrence connection $\mu_0(y) = \phi(y$,the elements $\mu_h(y), h \geq 1$ complete identification (y) of the confusing function $\mu(y)$: [4]

$$\mu_{m+1}(y) = \int_{\alpha}^{y} k(y,t)\,\mu_m(t)\,dt, \qquad , \quad m \geq 0. \tag{5}$$

Example (1)

To solve the following integral equation:

$$\mu(y) = 1 - \int_0^y \mu(t)dt. \tag{6}$$

where $\phi(y) = 1$ *and* $k(y,\ t) = -1$.

Decomposition series (4) is substituted into both sides of VIE (6) to produce [4]

$$\sum_{m=0}^{\infty} \mu_m(y) = 1 - \int_0^y \sum_{m=0}^{\infty} \mu_m(t)\,dt.$$

All terms not covered by the zeroth component are identified using the integral sign. As a result, we arrive at the recurrence connection shown below:

$$\mu_0(y) = 1$$

$$\mu_{n+1}(y) = -\int_0^y \mu_n(t)dt, \qquad s.t\ n \geq 0$$

$$\mu_1(y) = -\int_0^y \mu_0(t)\,dt = -\int_0^y 1dt = -y$$

$$\mu_2(y) = -\int_0^y \mu_1(t)\,dt = -\int_0^y -tdt = \frac{y^2}{2!}$$

$$\mu_3(y) = -\int_0^y \mu_2(t)\,dt = -\int_0^y \frac{t^2}{2!}dt = -\frac{y^3}{3!}$$

$$\mu_4(y) = -\int_0^y \mu_3(t)\,dt = -\int_0^y -\frac{t^3}{3!}dt = \frac{y^4}{4!}$$

The series solution is: $\mu(y) = 1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \frac{y^4}{4!} + \dots = e^{-y}$

2.2 THE SUCCESSIVE APPROXIMATION METHOD

The Picard iteration approach is commonly known as the sequential approximation method. This approach finds successive solution approximations by beginning with a first hunch, known as the zeroth approximation, to solve any problem. As seen, any chosen real-valued function will serve as the zeroth approximation and be used to calculate the other approximations in a recurrence. The recurrence relation is introduced through the successive approximations method: [4]

$$\mu_m(y) = \phi(y) + \int_{\alpha}^{y} k(y,t)\mu_{m-1}(t)\,dt\,, \qquad m \geq 1 \tag{7}$$

where any selected zeroth approximation of a real-valued function $\mu_0(y)$ is possible. We always make educated guesses for $\mu_0(y)$, typically choosing $(0, 1, y)$. We can get numerous subsequent approximations $\mu_n(y)$, $n \geq 1$ as follows: [9]

$$\mu_1(y) = \phi(y) + \int_{\alpha}^{y} k(y,t)\mu_0(t)\,dt$$
$$\mu_2(y) = \phi(y) + \int_{\alpha}^{y} k(y,t)\mu_1(t)\,dt$$
$$\mu_3(y) = \phi(y) + \int_{\alpha}^{y} k(y,t)\mu_2(t)\,dt$$
$$\vdots$$
$$\mu_m(y) = \phi(y) + \int_{\alpha}^{y} k(y,t)\mu_{m-1}(t)\,dt.$$

The following example will serve as an example of the sequential approximation approach or the Picard iteration method. [1]

Example (2)

To solve the following Volterra integral equation, use the successive approximation technique:

$$\mu(y) = -1 + e^y + \frac{1}{2}y^2e^y - \frac{1}{2}\int_0^y t\mu(t)\,dt. \tag{8}$$

We choose $\mu_0(y) = 0$ as the zeroth approximation for $\mu_0(y)$.

The iteration formula is then applied: [4]

$$\mu_{m+1}(y) = -1 + e^y + \frac{1}{2}y^2e^y - \frac{1}{2}\int_0^y t\mu_m(t)\,dt \qquad , m \geq 0 \tag{9}$$

When $\mu_0(y)$ is substituted in the equation above, we get:

$$\mu_1(y) = -1 + e^y + \tfrac{1}{2}y^2e^y$$
$$\mu_2(y) = -3 + \tfrac{1}{4}y^2 + e^y\left(3 - 2y + \tfrac{5}{4}y^2 - \tfrac{1}{4}y^3\right)$$
$$\mu_3(y) = y\left(1 - y + \tfrac{y^2}{2!} - \tfrac{y^3}{3!} + \tfrac{y^4}{4!} + \ldots\right)$$
$$\vdots$$
$$\mu_{m+1}(y) = y\left(1 - y + \tfrac{y^2}{2!} - \tfrac{y^3}{3!} + \tfrac{y^4}{4!} + \ldots\right) = ye^{-y}.$$

2.3 THE TECHNIQUE OF LAPLACE TRANSFORMATION

The Laplace transformation technique can be used to solve integral equations if the kernel depends on the difference $(y-t)$. The following outcomes are attained by transforming both sides of VIEs with the Laplace transform: [9]

$$\mu(z) = \phi(z) + K(z)\mu(z)\,. \tag{10}$$

Where $\mu(z) = L(\mu(y)\}\,,\quad K(z) = L(K(y,t)\}\,,\quad \phi(z) = L(\phi(y)\}\,.$

By using the universal of the Laplace transform of the problem $\mu(y)$: [10]

$\mu(z) = \frac{\phi(z)}{1-K(z)},\quad K(z) \neq 0.$

Then, we discover that $\mu(y) = L^{-1}\left(\frac{\phi(z)}{1-K(z)}\right\}.$

The following example will demonstrate this method. [4]

Example (3)

The following Volterra integral equation needs to be solved:

$$\mu (y) = 1 - \int_0^y (y - t)\mu (t)\, dt. \tag{11}$$

Transforming both sides of the equation using the Laplace method above in the case where $\phi(y) = 1$ and $k(y, t) = (y - t)$ results in: [4]

$L (\mu (y)\} = L (1) - L ((y)\}\ L\{(\mu)\}$

So that

$\mu (z) = \frac{1}{z} - \frac{1}{z^2}\mu (z)$

$\mu (z) = \frac{z}{1+z^2} \implies \mu (y) = \phi^{-1} \left(\frac{z}{1+z^2}\right]$

The exact answer to the equation above is $\mu(y) = cosy$.

2.4 THE MODIFIED METHOD OF DECOMPOSITION

The A domain decomposition method breaks the solution into endless components. It is possible to calculate the Volterra integral equation's inhomogeneous part, $\phi(y)$, as follows:

$$\mu (y) = \phi (y) + \lambda \int_0^y K (y, t)\mu (t)\, dt. \tag{12}$$

The evaluation of the components $\mu_h,\ h \geq 0$ involves time-consuming work; if $\phi(y)$ is a function that mixes polynomials, trigonometric functions, hyperbolic functions, and other functions, it must do so. [5] The updated decomposition method will simplify the calculation and hasten the convergence of the series solution even more. Everywhere appropriate, the modified decomposition approach will solve all differential and integral equations, regardless of order. It is crucial to remember that the improved decomposition strategy, which relies on dividing the function $\phi(y)$ into two parts cannot be applied if the function only has one component. [11] We can characterize the tactic more precisely if we keep in mind that the traditional A domain decomposition approach allows for the recurrence relation's use: [4]

$\mu_0 (y) = \phi (y)$.

$$\mu_{n+1} (y) = \lambda \int_0^y K (y, t)\mu_n (t)\, dt \ \ ,\ n \geq 0. \tag{13}$$

If the formula for the answer $\mu(y)$ is an endless sum of previously determined parts:

$$\mu (y) = \sum_{m=0}^{\infty} \mu_m (y). \qquad (14) \tag{14}$$

Evaluation of the $\mu_m (y) , m \geq 0$ components is straightforward. The recurrence relation is slightly altered by the modified decomposition method, which makes it simpler and quicker to identify the components of $\mu(y)$. [12] In certain situations, it is possible to set the function $\phi(y)$ as the combination of the following two partial functions: $\phi_1(y)$ *and* $\phi_2(y)$. In another way, we can get:

$$\phi (y) = \phi_1 (y) + \phi_2 (y) . \tag{15}$$

We change how the recurrence relation is formed qualitatively. We determine the zeroth component to reduce the number of calculations. $\mu_0(y)$, by a piece of $\phi\ (y)$, either $\phi_1 (y)\, or\ \phi_2 (y)\, of(y)$. Among other phrases, the component $\mu_1(y)$ can be supplemented with the other portion of $\phi(y)$. The modified decomposition approach, in other terms, introduces the modified recurrence relation: [13]

$\mu_0(y) = \phi_1(y),$

$$\mu_1 (y) = \phi_2 (y) + \lambda \int_0^y K (y, t)\mu_0 (t)\, dt. \tag{16}$$

$$\mu_{n+1} (y) = \lambda \int_0^y K (y, t)\mu_n (t)\, dt \ \ ,\ n \geq 0.$$

The production of the first two components, $\mu_0(y)$ *and* $\mu_1(y)$, alone accounts for the difference between the standard and modified recurrence relation, as demonstrated by this. In the two recurrence relations, the additional elements $\mu_h,\ h \geq 2$ stays constant. Although the variation in how $\mu_0(y)$ *and* $\mu_1(y)$ are formed is small, significantly impacting the speed at which the solution converges and the amount of computational work required. Additionally, changing the number of words in $\phi_1(y)$ has an impact on all the components, not just $\mu_1(y)$. Numerous research studies supported this conclusion. [14] Here, two crucial points regarding the changed technique can be stated. First, by carefully choosing the functions $\phi_1(y)$ *and* $\phi_2(y)$, the exact answer $\mu(y)$ can often be found with a minimal number of repetitions, sometimes even with just two components being evaluated. Only the correct selection of $\phi_1(y)$ *and* $\phi_2(y)$ can ensure the effectiveness of this adjustment, and this decision can only be determined through trials. [15] There has not yet been discovered a rule that could aid in the proper selection of $\phi_1(y)$ *and* $\phi_2(y)$. Second, the conventional decomposition approach can be applied if $\phi(y)$ has only one term. It is important to note that both linear and nonlinear equations and Volterra and Fredholm integral equations will be solved using the modified decomposition method. The discussion of the ensuing instances will explain the updated decomposition method. [16]

Example (4)

Apply the Modified Decomposition Method to solve the following:

$$\mu(y) = siny + (e - e^{cosy}) - \int_0^y e^{cost}\mu(t)\,dt. \tag{17}$$

We initially divide the provided $\phi(y)$ into: [4]

$$\phi(y) = siny + (e - e^{cosy}). \tag{18}$$

Divided into two parts:

$$\phi_1(y) = siny, \qquad \phi_2(y) = e - e^{cosy}. \tag{19}$$

Next, we use the Modified recurrence formula above, which gives us the following:

$\mu_0(y) = \phi_1(y) = siny,$

$$\mu_1(y) = (e - e^{cosy}) - \int_0^y e^{cost}\mu_0(t)\,dt = 0. \qquad (20) \tag{20}$$

$$\mu_{n+1}(y) = -\int_0^y K(y,t)\mu_n(t)\,dt = 0\ ,\ n \geq 0.$$

Each element of $\mu_h, h \geq 1$ must be 0. The formula, then, gives the precise answer:

$\mu(y) = sin(y).$

2.5 THE METHOD OF THE SERIES SOLUTION

If a natural process $\mu(y)$ has derivatives of all orders, then it is said to be analytical if anywhere in its range, the Taylor series:

$$\mu(y) = \sum_{n=0}^{m} \frac{\phi^n(\beta)}{n!}(y-\beta)^n. \tag{21}$$

Converges to $\phi(y)$ in the area around β. For ease of writing, the Taylor series at $y = 0$ has the following general form: [4]

$$\mu(y) = \sum_{m=0}^{\infty} \alpha_m y^m. \tag{22}$$

In this section, we will go through a helpful technique, the Taylor series, for analytic functions mainly used for solving Volterra integral equations. We will suppose that the Volterra integral equation's $\mu(y)$ solution:

$$\mu(y) = \phi(y) + \lambda\int_0^y K(y,t)\mu(t)\,dt. \tag{23}$$

It has a Taylor series of the kind provided in (22) since it is analytical, and the coefficients will be determined repeatedly in this series. Inputting (22) on both sides of (23) results in the following: [1]

$$\sum_{m=0}^{\infty} \alpha_m y^m = T(\phi(y)) + \lambda \int_0^y K(y,t)\left(\sum_{m=0}^{\infty} \alpha_m t^m\right) dt. \quad (24)$$

alternatively, we can use the following:

$$\alpha_0 + \alpha_1 y + \alpha_2 y^2 + \cdots = T(\phi(y)) + \lambda \int_0^y K(y,t)\left(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \ldots\right) dt. \quad (25)$$

Where the Taylor series for $\phi(y)$ is $T\ (\phi(y))$. The equation for integrals (23) will be transformed into a conventional integral in (24) or (25), where the form terms $t^m, m \geq 0$ being integrated with a place of the unidentified function $\mu\ (y)$. It is significant to remember that because we are looking for a series solution, Taylor expansions for any functions contained in $\phi(y)$, including trigonometric and exponential functions, etc., should be employed. [6]

The integral's right side is first integrated into (24) or (25), then the coefficients of similar powers of y are gathered. To obtain a recurrence relation in $\alpha_h, h \geq\ 0$, we equalize the coefficients on both sides of the resulting equation comparable to y-powers. The coefficients $\alpha_h, h \geq 0$ can be entirely determined by solving the recurrence relation. The series solution is immediately obtained by putting the derived coefficients into the after determining the coefficients $\alpha_h, h \geq 0$ (22). If there is a precise solution, it may be possible to find it. The derived series can be applied numerically without an exact answer. In this scenario, our accuracy level increases as we assess more phrases. [17]

Example (5)

Apply the series method to solve:

$$\mu(y) = 2e^y - 2 - y + \int_0^y (y-t)\mu(t)\, dt. \quad (26)$$

Following the same procedure as before, we will take a few words from the Taylor series to determine the following: [1]

$$\alpha_0 + \alpha_1 y + \alpha_2 y^2 + \alpha_3 y^3 + \alpha_4 y^4 + \cdots = y + y^2 + \frac{1}{3}y^3 + \frac{1}{12}y^4 + \cdots + \int_0^y (y-t)\left(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \ldots\right) dt. \quad (27)$$

By combining the right side and gathering similar phrases to y, we discover:

$$\alpha_0 + \alpha_1 y + \alpha_2 y^2 + \alpha_3 y^3 + \alpha_4 y^4 + \ldots = y + \left(1 + \frac{1}{2}\alpha_0\right) y^2 + \left(\frac{1}{3} + \frac{1}{6}\alpha_1\right)$$

In (27), According to equating the coefficients of similar powers of y,

$$\alpha_0 = 0, \quad \alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = \frac{1}{2!}, \quad \alpha_4 = \frac{1}{3!} \quad (28)$$

And generally:

$$\alpha_m = \frac{1}{m!}, \quad m \geq 1, \quad \alpha_0 = 0. \quad (29)$$

The series form of the solution is as follows:

$$\mu(y) = y\left(1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \frac{1}{4!}y^4 + \ldots\right). \quad (30)$$

It leads to the precise answer:

$$\mu(y) = ye^y. \quad (31)$$

3. CONCLUSION

From the previous examples, we conclude the following:

This paper discussed various analytic methods for solving Second order Volterra integral equations—first, domain reduction. The hypothesis determines the order. Picard iteration uses successive approximation for the second way. The third way used is Laplace trans-formation. Fourth is changed decomposition. The Taylor series determines the fifth way, the series method. Variational iteration solves the VIE. Examples demonstrate these ways. .

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